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ON INSTANTONS ON NEARLY KÄHLER 6-MANIFOLDS*

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Abstract. We study ω-instantons on nearly Kähler 6-manifolds. These are defined as connections A whose curvatures F satisfy $*F = -\omega \wedge F$. First, we show these connections enjoy nice properties: they are Yang-Mills and variational. Second, we discuss their relation with instantons over the G_2 cones. Third, we derive a Weitzenböck formula for the infinitesimal deformation and derive some rigidity results. Fourth, we construct some $SO(4)$-invariant examples over open sets of S^6.

Key words. Nearly Kähler, Yang-Mills connections, Instantons.

AMS subject classifications. Primary 53C38

Introduction. The notion of anti-self-dual instantons plays an important role in Donaldson's theory of 4-manifolds ([7]). This concept has been generalized to higher dimensions (e.g., [8] and [11]). To motivate the generalization, we first recall the 4-dimensional theory.

Suppose M is an oriented 4-dimensional Riemannian 4-manifold. It is well known that the space of 2-forms splits into self-dual and anti-self-dual parts, corresponding respectively to ± 1-eigenspaces of Hodge $*$ operator. A connection A on a certain principal bundle over M is said to be an *anti-self-dual* instanton if its curvature F, when viewed as a vector-bundle valued two-form, satisfies $*F = -F$. Of course, this definition does not generalize directly to higher dimensions. If, moreover, M is almost Hermitian, i.e., endowed with an almost complex structure compatible with the Riemannian structure, we can formulate the notion in another way. This is based on the observation that anti-self-dual 2-forms are exactly ω-trace free $(1,1)$-forms. Thus, in the almost Hermitian case, we can equally define anti-self-dual instantons to be those connections A satisfying

$$F^{2,0} = \mathrm{tr}_\omega F = 0. \tag{1}$$

The latter description obviously allows generalizations to higher dimensional almost Hermitian manifolds. We will also call connections satisfying (1) *pseudo-Hermitian-Yang-Mills* by slight abuse of terminology (compare [3], for example).

When the dimension is 6, we can formulate (1) in yet another way. Notice that the operator $*(\omega \wedge \cdot)$ maps the space of two forms into itself. It can also be shown that the space of ω-trace free $(1,1)$-forms is exactly the -1 eigenspace of $*(\omega \wedge \cdot)$. Thus, we can rewrite the equation (1) as

$$\omega \wedge F = - * F. \tag{2}$$

For this reason, we also call pseudo-Hermitian-Yang-Mills connections ω-anti-self-dual instantons.

Now, (2) makes sense in even more general contexts. Suppose that M is endowed with an $n-4$ form Ω. Then the operator $*(\Omega \wedge \cdot)$ maps 2-forms into 2-forms. We

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