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HARMONIC DIFFEOMORPHISMS BETWEEN COMPLETE RIEMANN SURFACES OF NEGATIVE CURVATURE*

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Abstract. Consider two open Riemannian surfaces (M^2, g_0), (M^2, g_1), a curve $\{g_t\}_{0\le t\le 1}$ in the Sobolev topology, $K_{g_t} \equiv -1$, $\inf \sigma(\Delta_0(g_t)) > 0$, $r_{inj}(g_t) > 0$, $0 \le t \le 1$. We prove that there exists a unique harmonic diffeomorphism $f_1 : (M^2, g_0) \to (M^2, g_1)$ which is moreover isotopic by harmonic diffeomorphisms to $\mathrm{id} : (M^2, g_0) \to (M^2, g_0)$ in the unit component $\mathcal{D}_0^{r+1}$ of the completed diffeomorphism group $\mathcal{D}^{r+1}$. This has application in Teichmüler theory for open surfaces.

Key words. Harmonic map, open surfaces, Teichmüler space, diffeomorphism.

AMS subject classifications. 58D27, 58E20

1. Introduction. In [5] the second author introduced Teichmüller spaces for open manifolds (i.e., noncompact complete manifolds) as follows. Let (M^n, g) be an open Riemannian manifold with bounded geometry, the Sobolev completed space $\mathcal{M}^{2,r}(I, B_k)$ (c.f. [5]) splits into its arc components $\mathcal{M}^{2,r}(I, B_k) = \sum_{i\in I} comp^{2,r}(g_i)$. We considered the case $n = 2$, i.e. (M^2, g_0) with the sectional curvature $K_{g_0} \equiv -1$, the lowest spectral value $\inf \sigma(\Delta_0(g_0)) > 0$, the injectivity radius $r_{inj}(g_0) > 0$ and its arc component $comp^{2,r}(g_0)$. We defined a completed space $\mathcal{P}^r(g_0)$ of positive conformal factors. $\mathcal{P}^r = \sum_i comp(e^{u_i})$, and $comp(1) \subset \mathcal{P}^r(g_0)$ is an invariant of $comp(g_0)$. $comp(1)$ acts on $comp(g_0)$. We proved that under the above assumptions there exists for any $g \in comp(g_0)$ a unique $e^u \in comp(1) \subset \mathcal{P}^r(g_0)$ such that $K_{e^u g} \equiv -1$. Thereafter we defined as Teichmüller space of $comp^{2,r}(g_0)$ the space

$$\mathcal{T}^r(comp(g_0)) := comp(1)\backslash comp^{2,r}(g_0)/\mathcal{D}_0^{r+1}.$$

If M^2 would be closed, then $\mathcal{M}^{2,r}(I, B_k)$ consists of only one component, and we come back to the classical Teichmüller space.

In the open case, we would be interested in the topological structure of $\mathcal{T}^r(comp(g_0))$. It follows from the slice theorem in [7] that $\mathcal{T}^r(comp(g_0))$ is a Hilbert manifold. One canonical method to get some insight into the topological structure is the construction of a Morse function. Assume that for any $g_1 \in comp^{2,r}(g_0)$ with $K_{g_1} \equiv -1$ there exists a unique harmonic diffeomorphism $f : (M^2, g_0) \to (M^2, g_1)$ in $\mathcal{D}_0^{2,r+1}$, then we prove in a forthcoming paper that in fact

$$\phi(g_1) := \int_{M^2} [e(f : (M^2, g_0) \to (M^2, g_1)) - e(\mathrm{id} : (M^2, g_0) \to (M^2, g_0))] d\mathrm{vol}_x(g_0),$$

where $e(h)$ is the energy density of h, defines a Morse function on $\mathcal{T}^r(comp(g_0))$.

Hence we have to assure the assumption above. This is the content of this paper and our

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