ASIAN J. MATH.
Vol. 13, No. 4, pp. 465–472, December 2009

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DESINGULARIZATION AND SINGULARITIES OF SOME MODULI SCHEME OF SHEAVES ON A SURFACE*

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Abstract. Let X be a nonsingular projective surface over $\mathbb{C}$, and H_- and H_+ be ample line bundles on X in adjacent chamber of type (c_1, c_2). Let $0 < a_- < a_+ < 1$ be adjacent minichambers, which are defined from H_- and H_+, such that the moduli scheme $M(H_-)$ of rank-two a_--stable sheaves with Chern classes (c_1, c_2) is non-singular. We shall construct a desingularization of $M(a_+)$ by using $M(a_-)$. As an application, we study whether singularities of $M(a_+)$ are terminal or not in some cases where X is ruled or elliptic.

Key words. Moduli scheme of stable sheaves on a surface, singularities, desingularization.

AMS subject classifications. Primary 14J60; Secondary 14D20

1. Introduction. Let X be a projective non-singular surface over $\mathbb{C}$, H an ample line bundle on X. Denote by $M(H)$ the coarse moduli scheme of rank-two H-stable sheaves with fixed Chern class $(c_1, c_2) \in \mathrm{NS}(X) \times \mathbb{Z}$. In this paper we think about singularities and desingularization of $M(H)$ from the view of wall-crossing problem of H and $M(H)$.

Let H_- and H_+ be ample line bundles on X separated by only one wall of type (c_1, c_2). For a parameter $a \in (0, 1)$, one can define the a-stability of sheaves in such a way that a-stability of sheaves with fixed Chern class equals H_--stability (resp. H_+-stability) if a is sufficiently close to 0 (resp. 1) , and there is a coarse moduli scheme $M(a)$ of rank-two a-stable sheaves with Chern classes (c_1, c_2). Let a_- and $a_+ \in (0, 1)$ be parameters which are separated by only one miniwall. Assume $M_- = M(a_-)$ is non-singular. One can find such a_- when X is ruled or elliptic. We construct a desingularization $\tilde{\pi}_+ : \tilde{M} \to M_+$ of $M_+ = M(a_+)$ by using M_- and wall-crossing methods, and apply it to consider whether singularities of M_+ are terminal or not when X is ruled or elliptic.

Let $\overline{M}(H)$ denote the Gieseker-Maruyama compactification of $M(H)$. By [10], when X is minimal and its Kodaira dimension is positive, $\overline{M}(H)$ has the nef canonical divisor if $\dim \overline{M}(H)$ equals its expected dimension and if H is sufficiently close to K_X. Thus, to understand minimal models of a moduli scheme of stable sheaves, it can be meaningful to study singularities on $M(H)$. As a problem to be solved, it is desirable to extend results in this article to the case where M_- is not necessarily non-singular but its singularities are terminal (Remark 2.5).

NOTATION. *For a k-scheme S, X_S is $X \times S$ and* $\mathrm{Coh}(X_S)$ *is the set of coherent sheaves on X_S. For $s \in S$ and $E_S \in \mathrm{Coh}(X_S)$, E_s means $E \otimes k(s)$. For E and $F \in \mathrm{Coh}(X)$,* $\mathrm{ext}^i(E, F) := \dim \mathrm{Ext}^i_X(E, F)$ *and* $\hom(E, F) = \dim \mathrm{Hom}_X(E, F)$. $\mathrm{Ext}^i_X(E, E)^0$ *indicates* $\mathrm{Ker}(\mathrm{tr} : \mathrm{Ext}^i(E, E) \to H^0(\mathcal{O}_X))$. *For $\eta \in \mathrm{NS}(X)$, we define $W^\eta \subset \mathrm{Amp}(X)$ by* $\{H \in \mathrm{Amp}(X) \mid H \cdot \eta = 0\}$.

2. Desingularization of M_+ by using M_- . We begin with background materials. Let H_- and H_+ be ample divisors lying in neighboring chambers of type $(c_1, c_2) \in \mathrm{NS}(X) \times \mathbb{Z}$, and H_0 an ample divisor in the wall W of type (c_1, c_2) which lies in the closure of chambers containing H_- and H_+ respectively. (Refer to [8] about

*Received January 9, 2009; accepted for publication October 22, 2009.
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