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COMPACTNESS OF THE MASSIVE SEIBERG-WITTEN EQUATION*

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In 1994, the Seiberg-Witten (SW) equation was introduced by Witten [W]. This system couples the anti-self-dual (ASD) equation for $U(1)$-con-nections with a harmonic spinor on four manifolds. By counting the number of its solutions, one can define the SW invariants. Conjecturally, the SW invariants are equivalent to the Donaldson invariants for four manifolds. Unlike the ASD equation for $SU(2)$-connections in the Donaldson theory, the SW equation has an amazing property which makes the SW theory much easier. Namely, the moduli space of solutions to the SW equation is *compact*.

In this article, we generalize the SW equation by allowing the spinor to have mass or energy level up to level n. We show that this massive SW equation also has compactness property. We will use the moduli space of mSWequation to define invariants for four manifolds in [LX].

To prove our compactness result, we again need to use the Weitzenböck formula and bootstrapping arguments as in the original SW theory. However new ingredients are needed here. These include a eigenvalue estimate by Vafa and Witten, a repeated use of the Weitzenböck formula and a control of individual eigencomponent of the spinor field.

1. Brief review of Seiberg-Witten theory. For any smooth compact closed oriented 4-manifold M, with given Riemannian metric and $spin^c$ structure, the Seiberg-Witten equation is defined for a $spin^c$ connection A and a positive spinor section ϕ,

$$D_A\phi = 0, \tag{1}$$

$$F_A^+ = \sigma(\phi), \tag{2}$$

in which $D_A : \Gamma(S^+) \to \Gamma(S^-)$ is the Dirac operator and

$$\sigma(\phi) = \phi\otimes\phi^* - \frac{|\phi|^2}{2}\mathrm{Id} \in \Gamma(\mathrm{ad}(S^+)) = \Omega_+^2(M; i\mathbb{R}) \tag{3}$$

can be identified with an imaginary valued self-dual 2-form.

The Seiberg-Witten moduli space is the quotient of the solution space of the Seiberg-Witten equation by the gauge group $Map(M, S^1)$. We have the following compactness theorem

THEOREM 1. *[KM][W] The Seiberg-Witten moduli space is compact.*

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