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AFFINE HERMITIAN-EINSTEIN METRICS*

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1. Introduction. A holomorphic vector bundle $E \to N$ over a compact Kähler manifold (N, ω) is called *stable* if every coherent holomorphic subsheaf F of E satisfies

$$0 < \text{rank } F < \text{rank } E \qquad \Longrightarrow \qquad \mu_\omega(F) < \mu_\omega(E),$$

where μ_ω is the ω-*slope* of the sheaf given by

$$\mu_\omega(E) = \frac{\deg_\omega(E)}{\text{rank } E} = \frac{\int_N c_1(E,h) \wedge \omega^{n-1}}{\text{rank } E}.$$

Here $c_1(E, h)$ is the first Chern form of E with respect to a Hermitian metric h. The famous theorem of Donaldson [7, 8] (for algebraic manifolds only) and Uhlenbeck-Yau [24, 25] says that an irreducible vector bundle $E \to N$ is ω-stable if and only if it admits a Hermitian-Einstein metric (i.e. a metric whose curvature, when the 2-form part is contracted with the metric on N, is a constant times the identity endomorphism on E). This correspondence between stable bundles and Hermitian-Einstein metrics is often called the Kobayashi-Hitchin correspondence.

An important generalization of this theorem is provided by Li-Yau [15] for complex manifolds (and subsequently due to Buchdahl by a different method for surfaces [3]). The major insight for this extension is the fact that the degree is well-defined as long as the Hermitian form ω on N satisfies only $\partial\bar\partial\omega^{n-1} = 0$. This is because

$$\deg_\omega(E) = \int_N c_1(E,h) \wedge \omega^{n-1}$$

and the difference of any two first Chern forms $c_1(E, h) - c_1(E, h')$ is $\partial\bar\partial$ of a function on N. But then Gauduchon has shown that such an ω exists in the conformal class of every Hermitian metric on N [9, 10]. (Such a metric on N is thus called a Gauduchon metric.) The book of Lübke-Teleman [18] is quite useful, in that it contains most of the relevant theory in one place.

An affine manifold is a real manifold M which admits a flat, torsion-free connection D on its tangent bundle. It is well known (see e.g. [20]) that M is an affine manifold if and only if M admits an affine atlas whose transition functions are locally constant elements of the affine group

$$\text{Aff}(n) = \{\Phi\colon \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi\colon x \mapsto Ax + b\}.$$

(In this case, geodesics of D are straight line segments in the coordinate patches of M.) The tangent bundle TM of an affine manifold admits a natural complex structure, and it is often fruitful to think of M as a real slice of a complex manifold. In particular,

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