A GENERALIZATION OF CHENG’S THEOREM

PETER LI† AND JIAPING WANG‡

Key words. Ricci curvature, bottom spectrum, weighted Poincare inequality.

AMS subject classifications. 53C42, 53C21

0. Introduction. In this paper, we prove a generalization of a theorem of S.Y. Cheng on the upper bound of the bottom of the \( L^2 \) spectrum for a complete Riemannian manifold. In [C], Cheng proved a comparison theorem for the first Dirichlet eigenvalue of a geodesic ball. By taking the radius of the ball to infinity, he obtained an estimate for the bottom of the \( L^2 \) spectrum. In particular, he showed that if \( M^n \) is an \( n \)-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below by \(-(n-1)K\) for some constant \( K > 0\), then the bottom of the \( L^2 \) spectrum, \( \lambda_1(M) \), is bounded by

\[
\lambda_1(M) \leq \frac{(n-1)^2K}{4}.
\]

This upper bound of \( \lambda_1(M) \) is sharp as it is achieved by the hyperbolic space form \( \mathbb{H}^n \). Observe that Cheng’s theorem can be stated in the following equivalent form.

**Cheng’s Theorem.** Let \( M^n \) be a complete Riemannian manifold of dimension \( n \). If \( \lambda_1(M) > 0 \) and there exists a constant \( A \geq 0 \) such that the Ricci curvature of \( M \) satisfies

\[
\text{Ric}_M \geq -A\lambda_1(M),
\]

then \( A \) must be bounded by

\[
A \geq \frac{4}{n-1}.
\]

In a previous paper [LW] of the authors, they consider complete Riemannian manifolds on which there is a nontrivial weight function \( \rho(x) \geq 0 \) for all \( x \in M \), such that, the weighted Poincaré inequality

\[
\int_M |\nabla \phi|^2 \, dV \geq \int_M \rho \phi^2 \, dV
\]

is valid for all functions \( \phi \in C_c^\infty(M) \). Note that if \( \lambda_1(M) > 0 \) then \( \lambda_1(M) \) can be used as a weight function by the variational characterization of \( \lambda_1(M) \), namely,

\[
\inf_{\phi \in C_c^\infty(M)} \frac{\int_M |\nabla \phi|^2 \, dV}{\int_M \phi^2 \, dV} = \lambda_1(M).
\]

With this point of view, a weight function \( \rho \) can be thought of as a pointwise generalization of \( \lambda_1(M) \). It was pointed out in [LW] that manifolds possessing a weighted Poincaré inequality is equivalent to being nonparabolic - those admitting a positive

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*Received February 20, 2008; accepted for publication June 18, 2008.
† Department of Mathematics, University of California, Irvine, CA 92697-3875, USA (pli@math.uci.edu). The author was partially supported by NSF grant DMS-0503735.
‡ School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA (jiaping@math.umn.edu). The author was partially supported by NSF grant DMS-0706706.