ASIAN J. MATH.
Vol. 12, No. 4, pp. 519–526, December 2008

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A GENERALIZATION OF CHENG'S THEOREM*

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Key words. Ricci curvature, bottom spectrum, weighted Poincare inequality.

AMS subject classifications. 53C42, 53C21

0. Introduction. In this paper, we prove a generalization of a theorem of S.Y. Cheng on the upper bound of the bottom of the L^2 spectrum for a complete Riemannian manifold. In [C], Cheng proved a comparison theorem for the first Dirichlet eigenvalue of a geodesic ball. By taking the radius of the ball to infinity, he obtained an estimate for the bottom of the L^2 spectrum. In particular, he showed that if M^n is an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below by $-(n-1)K$ for some constant $K > 0$, then the bottom of the L^2 spectrum, $\lambda_1(M)$, is bounded by

$$\lambda_1(M) \leq \frac{(n-1)^2 K}{4}.$$

This upper bound of $\lambda_1(M)$ is sharp as it is achieved by the hyperbolic space form $\mathbb{H}^n$. Observe that Cheng's theorem can be stated in the following equivalent form.

CHENG'S THEOREM. *Let M^n be a complete Riemannian manifold of dimension n. If $\lambda_1(M) > 0$ and there exists a constant $A \geq 0$ such that the Ricci curvature of M satisfies*

$$Ric_M \geq -A\lambda_1(M), \tag{0.1}$$

then A must be bounded by

$$A \geq \frac{4}{n-1}.$$

In a previous paper [LW] of the authors, they consider complete Riemannian manifolds on which there is a nontrivial weight function $\rho(x) \geq 0$ for all $x \in M$, such that, the weighted Poincaré inequality

$$\int_M |\nabla \phi|^2 \, dV \geq \int_M \rho \, \phi^2 \, dV$$

is valid for all functions $\phi \in C_c^\infty(M)$. Note that if $\lambda_1(M) > 0$ then $\lambda_1(M)$ can be used as a weight function by the variational characterization of $\lambda_1(M)$, namely,

$$\inf_{\phi \in C_c^\infty(M)} \frac{\int_M |\nabla \phi|^2 \, dV}{\int_M \phi^2 \, dV} = \lambda_1(M).$$

With this point of view, a weight function ρ can be thought of as a pointwise generalization of $\lambda_1(M)$. It was pointed out in [LW] that manifolds possessing a weighted Poincaré inequality is equivalent to being nonparabolic - those admitting a positive

* Received February 20, 2008; accepted for publication June 18, 2008.
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