ASIAN J. MATH.
Vol. 12, No. 3, pp. 391–420, September 2008

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RIEMANNIAN EXPONENTIAL MAPS OF THE DIFFEOMORPHISM GROUP OF $\mathbb{T}^2$*

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Abstract. We study the exponential maps induced by right-invariant weak Riemannian metrics of Sobolev type of order $k \geq 0$ on the Lie group of smooth, orientation preserving diffeomorphisms of the two dimensional torus. We prove that for $k \geq 1$, but not for $k = 0$, each of them defines a smooth Fréchet chart of the identity.

Key words. Geodesic exponential maps, diffeomorphism group of the torus

AMS subject classifications. 58D05, 58E10

1. Introduction. The aim of this paper is to contribute towards a theory of Riemannian geometry on infinite dimensional Lie groups. These groups have attracted a lot of attention since Arnold's seminal paper [1] on hydrodynamics – e.g. [12], [19], [24], [25]. As a case study we consider the Lie group $\mathcal{D}_+ = \mathcal{D}_+(\mathbb{T}^2)$ of orientation preserving C^∞-diffeomorphisms of the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The Lie algebra $T_{\mathtt{id}}\mathcal{D}_+$ of $\mathcal{D}_+$ is the space $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ of smooth vector fields on $\mathbb{T}^2$. We remark that $\mathcal{D}_+$ and its Lie algebra come up in hydrodynamics playing the role of configuration spaces for compressible and inviscid fluids on $\mathbb{T}^2$.

For any given $k \geq 0$, consider the scalar product $\langle \cdot, \cdot \rangle_k : C^\infty(\mathbb{T}^2, \mathbb{R}^2) \times C^\infty(\mathbb{T}^2, \mathbb{R}^2) \to \mathbb{R}$,

$$\langle u, v \rangle_k := \sum_{0 \leq j \leq k} \int_{\mathbb{T}^2} \langle (-\Delta)^j u, v \rangle dx$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^2$. It induces a C^1_F-smooth weak right-invariant Riemannian metric $\nu^{(k)}$ on $\mathcal{D}_+$,

$$\nu^{(k)}_\varphi(\xi, \eta) := \langle (d_{\mathtt{id}} R_\varphi)^{-1} \xi, (d_{\mathtt{id}} R_\varphi)^{-1} \eta \rangle_k, \quad \forall \varphi \in \mathcal{D}_+, \quad \text{and} \quad \forall \xi, \eta \in T_\varphi \mathcal{D}_+$$

where $R_\varphi : \mathcal{D}_+ \to \mathcal{D}_+$, $\psi \mapsto \psi \circ \varphi$ denotes the right translation by φ. The subscript F in C^1_F, refers to the calculus in Fréchet spaces – see Appendix A where we collect some definitions and notions of the calculus in Fréchet spaces. The metric $\nu^{(k)}$ being weak means that the topology induced by $\nu^{(k)}$ on the tangent space $T_\varphi\mathcal{D}_+$ at an arbitrary point φ in $\mathcal{D}_+$, is weaker than the Fréchet topology on $T_\varphi\mathcal{D}_+ \cong C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ – see e.g. [12].

DEFINITION 1.1. *For any given $T > 0$, a C^2_F-smooth curve $\varphi : [0, T] \to \mathcal{D}_+$, is called a geodesic for $\nu^{(k)}$, or $\nu^{(k)}$-geodesic for short, if it is a critical point of the*

*Received August 4, 2005; accepted for publication May 2, 2008.
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