ASIAN J. MATH.
Vol. 12, No. 3, pp. 321–344, September 2008

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DYNAMICS OF LINEAR AND AFFINE MAPS*

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Abstract. The well-known theory of the "rational canonical form of an operator" describes the invariant factors, or equivalently, elementary divisors, as a complete set of invariants of a similarity class of an operator on a finite-dimensional vector space $\mathbb{V}$ over a given field $\mathbb{F}$. A finer part of the theory is the contribution by Frobenius dealing with the structure of the centralizer of an operator. The viewpoint is that of finitely generated modules over a PID, cf. for example [8], ch. 3. In this paper we approach the issue from a "dynamic" viewpoint. We also extend the theory to affine maps. The formulation is in terms of the action of the general linear group $GL(n)$, resp. the group of invertible affine maps $GA(n)$, on the semigroup of all linear, resp. affine, maps by conjugacy. The theory of rational canonical forms is connected with the orbits, and the Frobenius' theory with the orbit-classes, of the action of $GL(n)$ on the semigroup of linear maps. We describe a parametrization of orbits and orbit-classes of both $GL(n)$- and $GA(n)$-actions, and also provide a parametrization of all linear and affine maps themselves, which is independent of the choices of linear or affine coordinate systems, cf. sections 7, 8, 9. An important ingredient in these parametrizations is a certain flag. For a linear map T on $\mathbb{V}$, let $Z_L(T)$ denote its centralizer associative $\mathbb{F}$-algebra, and $Z_L(T)^*$ the multiplicative group of invertible elements in $Z_L(T)$. In this situation, we associate a canonical, maximal, $Z_L(T)$-invariant flag, and precisely describe the orbits of $Z_L(T)^*$ on $\mathbb{V}$, cf. section 3. The classical theory uses only invariance under T, i.e. $\mathbb{V}$ is considered only as a module over $\mathbb{F}[T]$. The finer invariance under $Z_L(T)$, i.e. considering $\mathbb{V}$ as a module over $Z_L(T)$, makes the construction of the flag *canonical*. We believe that this flag has not appeared before in this classical subject. Using this approach, we strengthen the classical theory in a number of ways.

Key words. Rational canonical form, centraliser, dynamical types

AMS subject classifications. 12E99, 15A01, 15A02, 15A21, 37C45

1. Introduction. Let $\mathbb{F}$ be a field, and $\mathbb{V}$ an n-dimensional vector space over $\mathbb{F}$. Let $L(\mathbb{V})$ denote the set of all linear maps from $\mathbb{V}$ to $\mathbb{V}$. Underlying $\mathbb{V}$ there is the affine space $\mathbb{A}$. Intuitively, $\mathbb{A}$ has no distinguished base-point which one can call as the"zero", or the "origin". However there is a well-defined notion of "difference of points". When we distinguish a base-point O, and call it the zero, then there is a well-defined notion of addition, making $\mathbb{A}$ into a vector space. An *affine map* of $\mathbb{A}$ is a map $(A, v) : \mathbb{V} \to \mathbb{V}$ of the form $(A, v)(x) = Ax + v$, where A is in $L(\mathbb{V})$, and x, b are in V. Then

$$(A_1, v_1) \circ (A_2, v_2) = (A_1 \circ A_2, A_1 v_2 + v_1). \tag{1.1}$$

This formula shows that $A(\mathbb{V})$ is a semigroup with identity under composition, and $L(\mathbb{V})$ is a sub-semigroup of $A(\mathbb{V})$.

It is important to note that the representation (A, v) depends on the choice of the base-point. However the semigroup of affine maps, and the form of an affine map is independent of this choice. Indeed, let O be a base-point making $\mathbb{A}$ into a vector space $\mathbb{V}$. Let P be another point of $\mathbb{A}$ with the associated vector a. Let x resp y be vector representations of a point Q w.r.t. base-points O and P. Then $y = x - a$. Let f be an affine map of the form (A, v) in the x-representation, and $f(Q) = R$. Then the x-representation of R is $Ax + v = Ay + Aa + v$. So the y-representation of R is $Ay + Aa + v - a = Ay + w$, where $w = (A - I)a + v$. Hence the y-representation of f is (A, w). The maps induced by the action of the group $(\mathbb{V}, +)$ on $\mathbb{V}$, called the

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