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PALINDROMIC BRAIDS*

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Abstract. The braid group B_n, endowed with Artin's presentation, admits an antiautomorphism $B_n \to B_n$, such that $v \mapsto \overline{v}$ is defined by reading braids in reverse order (from right to left instead of left to right). We prove that the map $B_n \to B_n$, $v \mapsto v\overline{v}$ is injective. We also give some consequences arising due to this injectivity.

Key words. Braid, palindrome, Garside, Jacquemard

AMS subject classifications. 11E81, 11E39

1. Introduction. Let $n \geq 2$. Any free group F_{n-1} on $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ supports the antiautomorphism $rev : w \mapsto \overline{w}$ defined by

$$\sigma_{i_1}^{\alpha_1} \cdots \sigma_{i_r}^{\alpha_r} \mapsto \sigma_{i_r}^{\alpha_r} \cdots \sigma_{i_1}^{\alpha_1},$$

which reverses the order of the word w with respect to the prescribed set of generators. It follows that any group G presented by generators and relations admits such an antiautomorphism rev. The elements of G which are order-reversing invariant are called *palindromic*. In this paper, we consider palindromic elements of Artin's braid group B_n, equipped with Artin's presentation, which will be called *palindromic braids*. Artin's presentation of the braid group B_n consists of $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } \quad |i-j| \geq 2, \tag{1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n-2. \tag{2}$$

We distinguish between two equivalence relations on the elements of the braid group. For two braid words a, b, we write $a = b$ to denote that a and b represent the same element in the group, and $a \equiv b$ to denote that a and b are actually the same element written letter by letter (i.e., $a \equiv b$ means that a and b are equal in the free group using only the generators of the braid group, with no relations).

Palindromic braids have a particularly nice geometric interpretation. Given a geometric braid β, denote by $\widehat{\beta}$ its closure into a link inside a fixed solid torus $D^2 \times S^1$.

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