Asian J. Math.
Vol. 12, No. 1, pp. 001–034, March 2008

GEOMETRIC HAMILTONIAN STRUCTURES ON FLAT SEMISIMPLE HOMOGENEOUS MANIFOLDS*

G. MARÍ BEFFA†

Abstract. In this paper we describe Poisson structures defined on the space of Serret-Frenet equations of curves in a flat homogeneous space G/H where G is semisimple. These structures are defined via Poisson reduction from Poisson brackets on $\mathcal{L}\mathfrak{g}^*$, the space of Loops in $\mathfrak{g}^*$. We also give conditions on invariant geometric evolution of curves in G/H which guarantee that the evolution induced on the differential invariants is Hamiltonian with respect to the most relevant of the Poisson brackets. Along the way we prove that differential invariants of curves in semisimple flat homogeneous spaces have order equal to 2 or higher, and we also establish the relationship between classical moving frames (a curve in the frame bundle) and group theoretical moving frames (equivariant G-valued maps on the jet space).

Key words. Invariant evolutions of curves, flat homogeneous spaces, Poisson brackets, differential invariants, completely integrable PDEs, moving frames

AMS subject classifications. Primary 37K; Secondary 53A55

1. Introduction. The subject of infinite dimensional Hamiltonian structures has applications to different branches in mathematics but its study has traditionally been an important component in the study of completely integrable systems. In fact, the majority of completely integrable systems of PDEs are Hamiltonian with respect to two different but compatible infinite dimensional Hamiltonian structures, that is, they are biHamiltonian. This property allows the generation of a recursion operator that produces an infinite sequence of preserved functionals.

The connection between classical differential geometry and completely integrable PDEs dates back to Liouville, Bianchi and Darboux ([Li], [Bi], [Da]), but it was after Hasimoto's work in the vortex filament flow evolution that the close relation between integrable PDEs and the evolution of curvature and torsion (rather than the curve flow itself) was clear. In fact, Hasimoto ([Ha]) proved that the vortex filament flow induces a completely integrable evolution on the curvature and torsion of the flow. In particular, the evolution of curvature and torsion was biHamiltonian. Langer and Perline pointed out in their papers on the subject (see [LP1], [LP2]) that the Hamiltonian structures that were used to integrate some of these systems were defined directly from the Euclidean geometry of the flow. This situation was known to exist not only in Riemannian geometry but also in projective geometry. In fact, the Schwarzian KdV equation

$$u_t = u_x S(u)$$

where $S(u) = \frac{u_{xxx}u_x - 3/2u_{xx}^2}{u_x^2}$ is the Schwarzian derivative of u, has been known to be a curve evolution inducing a KdV evolution on $S(u)$. The Schwarzian derivative $S(u)$ is the differential invariant of reparametrizations of the projective line (or curves in $\mathbb{RP}^1$). There are also $\mathrm{PSL}(n,\mathbb{R})$-invariant flows of curves in $\mathbb{RP}^n$ inducing generalized KdV equations on the projective differential invariants of the flow ([DS], [M2]). The Hamiltonian structures used to integrate them can be defined directly from the

*Received November 14, 2006; accepted for publication March 22, 2007.
†Mathematics Department, University of Wisconsin, Madison, Wisconsin 53706, USA (maribeff@bmath.wisc.edu).