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DISCRETIZATION OF VECTOR BUNDLES AND ROUGH LAPLACIAN*

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Abstract. Let $\mathcal{M}(m,\kappa,r_0)$ be the set of all compact connected m-dimensional manifolds (M,g) such that $Ricci(M,g) \geq -(m-1)\kappa g$ and $Inj(M,g) \geq r_0 > 0$. Let $\mathcal{E}(n,k_1,k_2)$ be the set of all Riemannian vector bundles (E,∇) of real rank n with $|R^E| \leq k_1$ and $|d^*R^E| \leq k_2$. For any vector bundle $E \in \mathcal{E}(n,k_1,k_2)$ with harmonic curvature or with complex rank one, over any $M \in \mathcal{M}(m,\kappa,r_0)$ and for any discretization X of M of mesh $0 < \varepsilon \leq \frac{1}{20}r_0$, we construct a canonical twisted Laplacian Δ_A and a potential V depending only on the local geometry of E and M such that we can compare uniformly the spectrum of the rough Laplacian $\overline{\Delta}$ associated to the connection of E and the spectrum of $\Delta_A + V$. We show that there exist constants c, $c' > 0$ depending only on the parameters of $\mathcal{M}(m,\kappa,r_0)$ and $\mathcal{E}(n,k_1,k_2)$ such that $c'\lambda_k(X,A,V) \leq \lambda_k(E) \leq c\lambda_k(X,A,V)$, where $\lambda_k(\cdot)$ denotes the k^{th} eigenvalue of the considered operators ($k \leq n|X|$). For flat vector bundles, we show that the potential is zero, Δ_A turns out to be a discrete magnetic Laplacian and we relate $\lambda_1(E)$ to the holonomy of E.

Key words. Connection, rough Laplacian, discrete magnetic Laplacian, Harper operator, eigenvalues, discretization, holonomy

AMS subject classifications. 58J50, 53C20

1. Introduction. In [22], we have shown that for a family of compact connected manifolds $\mathcal{M}(m,\kappa,r_0)$ with injectivity radius and Ricci curvature bounded below (i.e. $(M,g) \in \mathcal{M}(m,\kappa,r_0)$ if M is a compact connected m-dimensional Riemannian manifold with $Ricci(M,g) \geq -(m-1)\kappa g$ and $Inj(M,g) \geq r_0$), we can compare uniformly the spectrum of the Laplacian acting on functions with the spectrum of the combinatorial Laplacian acting on a graph with fixed mesh constructed on the manifolds. Indeed, we show that there exist positive constants c, c' depending on the parameters of the problem such that for any $M \in \mathcal{M}(m,\kappa,r_0)$ and any discretization X of M (with mesh $\varepsilon < \frac{1}{2}r_0$), the following holds

$$c'\lambda_k(X) \leq \lambda_k(M) \leq c\lambda_k(X) \tag{1.1}$$

for $k < |X|$, where $\lambda_k(\cdot)$ stands for the k^{th} eigenvalue of the considered Laplacian. This result generalizes in a natural way different works like [5], [6], [9] and [19] that were motivated either by the study of the relation between the fundamental group of a manifold and the spectrum of its finite coverings ([5], [6]) or by the relation between the spectrum of a manifold and its Cheeger isoperimetric constant ([9]) or by the existence of harmonic functions ([19]). More generally, the aim of the discretization is to have an understanding of the spectrum (a global invariant on the manifold) with a minimum of informations about the local geometry of the manifold.

Of course, the problem is interesting for differential operators other than the Laplacian and we may address the following question: does the same kind of comparison hold for other geometric differential operators such that the Laplacian acting on p-forms or the Dirac operator? Most of these operators may be expressed in terms of a connection Laplacian added with a curvature term. In this article, we investigate

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