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006

THE NUMBER OF RATIONAL CURVES ON K3 SURFACES*

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Abstract. Let X be a $K3$ surface with a primitive ample divisor H, and let $\beta = 2[H] \in H_2(X, \mathbf{Z})$. We calculate the Gromov-Witten type invariants n_β by virtue of Euler numbers of some moduli spaces of stable sheaves. Eventually, it verifies Yau-Zaslow formula in the non primitive class β.

Key words. Rational curve, K3 surface, stable sheaf, Euler number

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Introduction. Let X be a $K3$ surface with an ample divisor H, and let $C \in |H|$ be a reduced curve. By adjunction formula, the arithmetic genus of C is $g = \frac{1}{2}H^2 + 1$. Under the assumption that the homology class $[H] \in H_2(X, \mathbf{Z})$ is primitive, Yau and Zaslow [18] showed that the number of rational curves in the linear system $|H|$ is equal to the coefficient of q^g in the series

$$\begin{aligned}\frac{q}{\Delta(q)} = \prod_{k>0} \frac{1}{(1-q^k)^{24}} &= \sum_{d \geq 0} G_d q^d \\ &= 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + \cdots\end{aligned}$$

Here a multiplicity $e(\bar{J}C)$ is assigned to each rational curve C in the counting([1]).

In [5], Fantechi, Göttsche and van Straten gave an interpretation of the multiplicity $e(\bar{J}C)$. Let $M_{0,0}(X, [H])$ be the moduli space of genus zero stable maps $f : \mathbf{P}^1 \to X$ with $f_*([\mathbf{P}^1]) = [H] \in H_2(X, \mathbf{Z})$. $M_{0,0}(X, [H])$ is a zero dimensional scheme which is in general nonreduced. Let $\iota : C \hookrightarrow X$ be a rational curve in the class $[H]$, and $n : \mathbf{P}^1 \to C$ its normalization. Then $f = \iota \circ n : \mathbf{P}^1 \to X$ is a closed point of $M_{0,0}(X, [H])$ and $e(\bar{J}C)$ is equal to the multiplicity of $M_{0,0}(X, [H])$ at f.

There is another formulation and generalization of Yau and Zaslow's formula by virtue of Gromov-Witten invariants. For $K3$ surfaces, the usual genus 0 Gromov-Witten invariants vanish. To remedy this, one can use the notion of twistor family developed by Bryan and Leung in [2] provided that β is a primitive class. In general, there is an algebraic geometric approach proposed by Jun Li [11] using virtual moduli cycles. Roughly speaking, he defines Gromov-Witten type invariants $N_g(\beta)$ on $K3$ surfaces by modifying the usual tangent-obstruction complex. When β is primitive, these invariants coincide with those defined by twistor family. Geometrically, $N_g(\beta)$ can be thought as Gromov-Witten invariants of a one dimensional family of $K3$ surfaces, which actually count curves in the original surface. For the rigorous definitions, see [2],[11].

Bryan and Leung [2] proved a formula for $N_g(\beta)$ when β is primitive. Let $n_\beta = N_0(\beta)$. Then $n_\beta = G_d$ with $d = \frac{1}{2}\beta^2 + 1$. It recovers the formula of Yau and Zaslow. For a non primitive class β, the numbers $N_g(\beta)$ are still unknown. However, there is a conjectural formula for $N_0(\beta)$([11]). Using the notation n_β, it says

$$n_\beta = \sum_k \frac{1}{k^3} G_{\frac{1}{2}(\frac{\beta}{k})^2+1}$$

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