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BOUNDARY VALUE PROBLEMS FOR HOLOMORPHIC FUNCTIONS ON THE UPPER HALF-PLANE*

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Abstract. Let $\Pi \subseteq \mathbb{C}$ be the open upper half-plane and let $\{\gamma_z\}_{z\in\partial\Pi}$ be a smooth family of smooth Jordan curves in the complex plane $\mathbb{C}$ parametrized by the boundary of Π. Then there exists a smooth up to the boundary holomorphic function f on Π such that $f(z) \in \gamma_z$ for every $z \in \partial\Pi$. Similar result is also proved on an arbitrary bordered Riemann surface.

Key words. Boundary value problem, Riemann-Hilbert problem

AMS subject classifications. Primary 30E25, 35Q15

1. Introduction. Let $\Pi = \{z \in \mathbb{C}; \mathrm{Im(z)} > 0\}$ be the open upper half-plane and let $\{\gamma_z\}_{z\in\partial\Pi}$ be a smooth family of smooth Jordan curves in the complex plane parametrized by the boundary $\partial\Pi$ of Π, that is, there exists a function $\rho \in C^\infty(\partial\Pi\times\mathbb{C})$ such that

$$\gamma_z = \{w \in \mathbb{C}; \rho(z,w) = 0\}$$

and $\overline{\partial}_w\rho(z,w) \neq 0$ for every $z \in \partial\Pi$ and $w \in \gamma_z$. We are interested in the existence of solutions of the corresponding Riemann-Hilbert problem and we show the following theorem.

THEOREM 1.1. *Let $\{\gamma_z\}_{z\in\partial\Pi}$ be a smooth family of smooth Jordan curves in $\mathbb{C}$. Then there exists a smooth up to the boundary holomorphic function f on Π such that $f(z) \in \gamma_z$ for every $z \in \partial\Pi$.*

Using conformal equivalence between the open upper half-plane Π and the open unit disc Δ one gets the following equivalent statement.

THEOREM 1.2. *Let $\{\gamma_z\}_{z\in\partial\Delta\setminus\{1\}}$ be a smooth family of smooth Jordan curves in $\mathbb{C}$. Then there exists a smooth function f on $\overline{\Delta}\setminus\{1\}$, holomorphic on Δ, such that $f(z) \in \gamma_z$ for every $z \in \partial\Delta\setminus\{1\}$.*

Let $\{\gamma_z\}_{z\in\partial\Delta}$ be a smooth family of smooth Jordan curves in $\mathbb{C}$ parametrized by the **whole** boundary $\partial\Delta$ of Δ. By Theorem 1.2 there are no obstructions to the existence of a solution of the Riemann-Hilbert problem on the disc for the family of Jordan curves $\{\gamma_z\}_{z\in\partial\Delta}$ if we allow solutions to be "wild" at only one boundary point. On the other hand the existence of a smooth up to the boundary holomorphic function f on Δ such that $f(z) \in \gamma_z$ for **every** $z \in \partial\Delta$ is not always guaranteed. For example one can take

$$\rho(z,w) = |w-\overline{z}|^2 - r^2,$$

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