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NORMAL BUNDLES OF RATIONAL CURVES IN PROJECTIVE SPACES*

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Abstract. We determine the splitting (isomorphism) type of the normal bundle of a generic genus-0 curve with 1 or 2 components in $\mathbb{P}^n$, as well as the way the bundle deforms locally with a general deformation of the curve. We deduce an enumerative formula for divisorial loci of smooth rational curves whose normal bundle is of non-generic splitting type.

Key words. Rational curve, projective space, normal bundle, enumerative geometry, cohomology theory, parameter spaces

AMS subject classifications. 14N10, 14N05

Rational curves in projective space, being essentially the same thing as finite-dimensional vector spaces of rational functions in one variable, are among the most elementary and classical objects in Algebraic Geometry. In recent years it has become clear that suitable (compact) parameter spaces, say $R_{n,d}$, for rational curves of given degree d in $\mathbb{P}^n$, are of fundamental importance. Now the geometry of a moduli or parameter space like $R_{n,d}$ is closely related to 'modular' subvarieties, i.e. ones defined in terms of the (universal) family of curves (or other objects) that it parametrizes. There are, to be sure, various ways of defining modular subvarieties of $R_{n,d}$, for instance the much-studied *incidence* subvarieties, parametrizing curves incident to a given cycle in $\mathbb{P}^n$. Another type of modular subvarieties involves vector bundles. Namely, given a 'reasonable function' Φ assigning to a curve $C \in R_{n,d}$ a vector bundle E_C on C, a theorem of Grothendieck says we have a decomposition

$$E_C \simeq \bigoplus \mathcal{O}_C(k_i), k_1 \geq k_2, ...$$

where $\mathcal{O}_C(k)$ denotes the unique line bundle of degree k on C. The sequence $k. = k.(C)$, which is uniquely determined and called the *splitting type* of E_C, varies upper-semicontinuously, in an obvious sense, in terms of the vector bundle and hence for a good function Φ we get a stratification $R^{\Phi}_{n,d}(k.)$ of $R_{n,d}$ where the strata consist of the curves C with given sequence $k.(C)$.

One way to define an interesting, and reasonable, function Φ is to fix a vector bundle E on $\mathbb{P}^n$ and to set

$$E_C = E|_C.$$

The resulting stratification was studied in [R5] where we computed enumeratively its divisorial stratum. The main result of this paper is an analogous computation in the case where Φ is the 'normal bundle function', which assigns to a curve C its *normal bundle*

$$N_C = N_{C/\mathbb{P}^n}.$$

The splitting type $k.(C)$ of N_C, which we call the *normal type* of C, is a natural global numerical invariant of the embedding $C \subset \mathbb{P}^n$, perhaps the most fundamental such

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