ASIAN J. MATH.
Vol. 11, No. 4, pp. 535–554, December 2007

COASSOCIATIVE CONES RULED BY 2-PLANES*

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Abstract. It is shown that coassociative cones in $\mathbb{R}^7$ that are r-oriented and ruled by 2-planes are equivalent to CR-holomorphic curves in the oriented Grassmanian of 2-planes in $\mathbb{R}^7$. The geometry of these CR-holomorphic curves is studied and related to holomorphic curves in S^6. This leads to an equivalence between associative cones on one side and the coassociative cones whose second fundamental form has an O(2) symmetry on the other. It also provides a number of methods for explicitly constructing coassociative 4-folds. One method leads to a family of coassociative 4-folds whose members are neither cones nor are ruled by 2-planes. This family directly generalizes the original family of examples provided by Harvey and Lawson when they introduced coassociative geometry.

Key words. Differential geometry, Calibrations, Coassociative

AMS subject classifications. Primary 53C38

1. Introduction. In 1982 Harvey and Lawson introduced coassociative 4-folds in $\mathbb{R}^7$ as an example of a calibrated geometry [6]. Using the perspective of exterior differential systems they proved a local existence theorem. Harvey and Lawson offered a single finite dimensional family of explicit examples (see section 9). In 1985 Mashimo classified the coassociative cones whose links are homogeneous [15]. Then for almost twenty years the literature was quiet on the subject. The silence was broken in 2004 when techniques used for studying special Lagrangian geometry (which is another type of calibrated geometry) began to be applied to coassociative geometry [13], [7], [11]. This renewal of activity was motivated by Joyce's construction of compact manifolds with G_2 holonomy [9] and the appearance of coassociative geometry in M-theory [1].

In [13] Lotay studied coassociative 4-folds in $\mathbb{R}^7$ that are ruled by 2-planes. He gave a local existence result and presented a method for using a holomorphic vector field on a naturally associated Riemann surface to deform a coassociative cone that is ruled by 2-planes. His work is an extension of Joyce's work on ruled special Lagrangian 3-folds [10].

This article revisits the geometry of coassociative cones that are ruled by 2-planes using the methods introduced by Bryant in [3]. This perspective realizes the Riemann surface that appeared in [13] as a CR-holomorphic curve for a G_2-invariant CR-structure on $\tilde{G}(2,7)$, the oriented Grassmanian of 2-planes in $\mathbb{R}^7$. In fact, Proposition 7.2 asserts that r-oriented 2-ruled coassociative cones are equivalent to CR-holomorphic curves in $\tilde{G}(2,7)$.

Studying the geometry of CR-holomorphic curves leads to new methods for constructing coassociative cones that are ruled by 2-planes. These methods arise from the close relationship between CR-holomorphic curves in $\tilde{G}(2,7)$ and holomorphic curves[1] in S^6. This relationship can be interpreted as an equivalence between the associative cones and a certain family of coassociative cones. The null-torsion holomorphic curves in S^6 also form the backbone for a new family of coassociative 4-folds that directly generalizes the family introduced by Harvey and Lawson [6]. The generic coassocia-

*Received April 18, 2006; accepted for publication August 18, 2006.
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[1]Some people may prefer using the term "pseudo-holomorphic curves" since the almost complex structure on the ambient space is not integrable.