ASIAN J. MATH.
Vol. 11, No. 3, pp. 361–372, September 2007


# YAU'S PROBLEM ON A CHARACTERIZATION OF ROTATIONAL ELLIPSOIDS*

UDO SIMON†

**Abstract.** S.T. Yau stated the following problem:
*Assume that the Euclidean principal curvatures $k_1, k_2$ of a closed surface in Euclidean 3-space satisfy the relation $k_1 = ck_2^3$ for some real constant c. Is the surface a rotational ellipsoid?*
We give a proof for closed analytic surfaces and study related problems.

**Key words.** Characterization of quadrics, rotational ellipsoid, surface of revolution, principal curvatures, support functions

**AMS subject classifications.** 53C40, 53A05, 53A15, 53C45

**1. Introduction.** Let the principal curvatures $k_1, k_2$ of a surface in Euclidean 3-space $E^3$ satisfy a differentiable relation

$$W(k_1, k_2) = 0.$$

Such surfaces are called *Weingarten surfaces*. In [1] S.S. Chern generalized a theorem of D. Hilbert and proved:

THEOREM CHERN. *Consider a Weingarten ovaloid with the property that the principal curvature $k_1$ is a strictly decreasing function of $k_2$. Then the ovaloid is a sphere.*

On a rotational ellipsoid the principal curvatures satisfy the relation $k_1 = ck_2^3$ for some positive constant $c$. Chern used this as a counterexample in the sense that, for a characterization of spheres, one cannot modify the assumption "decreasing" to "increasing" in his Theorem. Chern's counterexample became of great importance in the study of relations between curvature functions (and also the support function) of compact hypersurfaces in Euclidean space.

In analogy to the characterization of spheres in terms of curvature functions there are also characterizations including the support function; such results are similar to the theorems about Weingarten surfaces with monotonicity properties; see e.g. [3], section 3.9. Following Chern's example, this led to a study of the support function on rotational surfaces and in particular on rotational ellipsoids; see see e.g. [3], p. 102. During the last decade, several authors proved additional results on the characterization of (hyper-)quadrics in terms of curvature and support functions. We recall the following result as an example; see section 8 in [6]:

THEOREM L-S-S-W. *Let $x : M \to E^{n+1}$ be a hyperovaloid. The following properties are equivalent:*
*(i) x is a hyperellipsoid with center at the origin.*
*(ii) The Gauß-Kronecker curvature satisfies the following eigenvalue equation in terms of the third fundamental form metric $g^* = III$ :*

$$\Delta^*(H_n^{\frac{2}{n+2}} - c) + 2(n+1)(H_n^{\frac{2}{n+2}} - c) = 0,$$

*Received December 22, 2005; accepted for publication April 3, 2006.
†Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, D-10623 Berlin, Germany (simon@math.tu-berlin.de). Partially supported by DFG-PI-158/4-4 and DFG-PI/158-4-5.