ASIAN J. MATH.
Vol. 11, No. 3, pp. 341–360, September 2007

001

LEGENDRIAN VARIETIES*

J. M. LANDSBERG† AND L. MANIVEL‡

Abstract. We investigate the geometry of Legendrian complex projective manifolds $X \subset \mathbb{P}V$. By definition, this means V is a complex vector space of dimension $2n+2$, endowed with a symplectic form, and the affine tangent space to X at each point is a maximal isotropic subspace. We establish basic facts about their geometry and exhibit examples of inhomogeneous smooth Legendrian varieties, the first examples of such in dimension greater than one.

Key words. Legendrian variety, Kummer surface, K3 surface, Chern class

AMS subject classifications. 14J25, 14J28, 14M99, 53D99

1. Introduction.

The initial motivation for this project stems from the study of the holonomy groups of Riemannian manifolds, where the only open case for existence of compact non-homogeneous examples is the quaternion-Kähler case. Thanks to work of Salamon, LeBrun and others (see, e.g., [27, 24]), the question is essentially equivalent to the existence of inhomogeneous contact Fano manifolds (so far none are known). Several people [30, 29] observed that the set of tangent directions to minimal degree rational curves through a general point of a contact Fano manifold is a Legendrian subvariety in its projective span. S. Kebekus [18, 19] then showed:

THEOREM 1. *Let Y be a smooth contact Fano manifold with Picard number one, not a projective space. Let y a general point of Y, and denote by $\mathcal{H}_y \subset \mathbb{P}T_yY$ the set of tangent directions to contact lines on Y passing through y. Then $X = \mathcal{H}_y$ is a smooth Legendrian variety in its linear span.*

Moreover, if at all points of Y the corresponding Legendrian variety is homogenous and equivariantly embedded, Hong [14] proved that Y itself must be homogeneous.

The homogeneous examples of contact Fano manifolds are as follows: let $\mathfrak{g}$ denote a complex simple Lie algebra and G its adjoint group. Then G has a unique closed orbit X_G^{ad} inside $\mathbb{P}\mathfrak{g}$, the projectivization of its Lie algebra – we call this variety the *adjoint variety* of G. It is also the projectivization of the minimal nilpotent orbit in $\mathfrak{g}$. The adjoint varieties are contact Fano manifolds, and the conjecture of Lebrun and Salamon is that there exists no other.

The set of lines passing through a given point of an adjoint variety is a smooth homogeneous Legendrian variety. We call these varieties, in these particular embeddings, the *subadjoint varieties*. Classical examples are the twisted cubic in $\mathbb{P}^3$ (coming from the adjoint variety of the exceptional group G_2), and the products $\mathbb{P}^1 \times \mathbb{Q}^n$ of a projective line with a smooth quadric of dimension $n \geq 1$ (coming from the adjoint varieties of the orthogonal groups). Note that the symplectic groups give empty subadjoint varieties. The adjoint varieties of the other exceptional groups give rise to a remarkable series of homogeneous varieties, which we called the *subexceptional series*: they constitute the third line of the geometric version of Freudenthal's magic square,

*Received March 14, 2006; accepted for publication April 3, 2006.
†Department of Mathematics, Mailstop 3368, Texas A&M University, College Station, TX 77843-3368, USA (jml@math.tamu.edu).
‡Institut Fourier, UMR 5582 du CNRS, Université Grenoble I, BP 74, 38402 Saint Martin d'Hères cedex, France (Laurent.Manivel@ujf-grenoble.fr).