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A UNIVERSAL METRIC FOR THE CANONICAL BUNDLE OF A HOLOMORPHIC FAMILY OF PROJECTIVE ALGEBRAIC MANIFOLDS*

DROR VAROLIN†

Dedicated to M. Salah Baouendi on the occasion of his 70th birthday

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1. Introduction. In his celebrated work [S-98, S-02], Siu proved that the plurigenera of any algebraic manifold are invariant in families. More precisely, let $\pi : \mathscr{X} \to \mathbb{D}$ be a holomorphic submersion (i.e., $d\pi$ is nowhere zero) from a complex manifold $\mathscr{X}$ to the unit disk $\mathbb{D}$, and assume that every fiber $\mathscr{X}_t := \pi^{-1}(t)$ is a compact projective manifold. Then for every $m \in \mathbb{N}$, the function $P_m : \mathbb{D} \to \mathbb{N}$ defined by $P_m(t) := h^0(\mathscr{X}_t, mK_{\mathscr{X}_t})$ is constant.

Siu's approach to the problem begins with the observation that the function P_m is upper semi-continuous. Thus in order to prove that P_m is continuous (hence constant) it suffices to show that given a global holomorphic section s of $mK_{\mathscr{X}_0}$, there is a family of global holomorphic sections s_t of $\mathscr{X}_t$, for all t in a neighborhood of 0, that varies holomorphically with t and satisfies $s_0 = s$.

To prove such an extension theorem, Siu establishes a generalization of the Ohsawa-Takegoshi Extension Theorem to the setting of complex submanifolds of a Kahler manifold having codimension 1 and cut out by a single, bounded holomorphic function. This theorem, which we will discuss below, requires the existence of a singular Hermitian metric on the ambient manifold having non-negative curvature current, with respect to which the section to be extended is L^2. Thus in the presence of the extension theorem, the approach reduces to construction of such a metric.

The case where the fibers $\mathscr{X}_t$ of our holomorphic family are of general type was treated in [S-98]. In this setting, Siu produced a single singular Hermitian metric $e^{-\kappa}$ for K_X so that every m-canonical section is L^2 with respect to $e^{-(m-1)\kappa}$.

However, in the case where the fibers $\mathscr{X}_t$ of our holomorphic family are assumed only to be algebraic, and not necessarily of general type, Siu's proof in [S-02] does not construct a single metric as in the case of general type. Instead, Siu constructs for every section s of $mK_{\mathscr{X}_0}$ a singular Hermitian metric for $mK_{\mathscr{X}}$ of non-negative curvature so that s is L^2 with respect to this metric.

DEFINITION. Let $\mathscr{X} \to \Delta$ be a holomorphic family of complex manifolds and $\mathscr{X}_0$ the cental fiber of $\mathscr{X}$. A universal canonical metric for the pair $(\mathscr{X}, \mathscr{X}_0)$ is a singular Hermitian metric $e^{-\kappa}$ for the canonical bundle $K_{\mathscr{X}}$ of $\mathscr{X}$ such that for every global holomorphic section $s \in H^0(\mathscr{X}_0, mK_{\mathscr{X}_0})$,

$$\int_{\mathscr{X}_0} |s|^2 e^{-(m-1)\kappa} < +\infty.$$

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†Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA (dror@math.sunysb.edu). Partially supported by an NSF grant.