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ON PROPER HARMONIC MAPS BETWEEN STRICTLY PSEUDOCONVEX DOMAINS WITH KÄHLER METRICS OF BERGMAN TYPE*

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Dedicated to Salah Baouendi for his seventieth birthday

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1. Introduction. Let M and N be Kähler manifolds with respective Kähler metrics $h = h_{i\bar{j}}dz_i \otimes d\bar{z}_j$ and $g = g_{\alpha\bar{\beta}}dw^\alpha \otimes d\overline{w}^\beta$, respectively. A map $u : M^m \to N^n$ is said to be harmonic if the tension field $\tau^s[u]$ satisfies

$$\tau^s[u] = \Delta_M u^s + \sum_{i,j=1}^{m} \sum_{t,\gamma=1}^{n} \Gamma^s_{t\gamma}\partial_i u^t \partial_{\bar{j}} u^\gamma h^{i\bar{j}} = 0 \ \text{ for } 1 \le s \le n, \tag{1.1}$$

where $(h^{i\bar{j}})^t$ is the inverse of the matrix $(h_{i\bar{j}})$, $\Delta_M = \sum_{i,j} h^{i\bar{j}}\partial_{i\bar{j}}$ and $\Gamma^s_{t\gamma}$ denote the Christoffel symbols of the Hermitian metric g on N. It follows from (1.1) that if u is holomorphic, then u must be harmonic. Thus, it is natural to ask under what circumstances a harmonic map is holomorphic or antiholomorphic. Under the assumption that both M and N are compact, Siu [31] demonstrated that if the curvature tensor of N is strongly negative and the rank of du is greater than or equal to four at a point of M, then a harmonic map u must be holomorphic or antiholomorphic. The proof follows from Siu's Bochner type identity together with the compactness assumption on M.

If M is a complete noncompact manifold of strongly negative curvature with infinite volume, the previous Bochner type identity technique fails and not much is known about the rigidity of u. In general, the answer to the above posed question is negative: one needs to add some natural conditions to the map such as being a proper map. Along this direction, when M and N are unit balls in $\mathbb{C}^n$ endowed with Bergman metrics (the simplest case of Kähler manifolds with strongly negative curvature) progress was made by Li and Ni in [25]. They showed that for $m > 1$, if $u : (B^m, h) \to (B^n, g)$ is a C^2 up to the boundary pluriharmonic proper map, where h and g are respective Bergman metrics on B^m and B^n, then u must be holomorphic or antiholomorphic. In addition to this, several other equivalent conditions were given (cf. [25]).

The main purpose of this paper is to use a similar approach to the one given in (cf. [25]) to generalize their theorem from unit balls to smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^m$ and $\mathbb{C}^n$ for $m > 1$ with more general metrics of Bergman type. More precisely, we consider two smoothly bounded strictly pseudoconvex domains Ω_m and Ω_n in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. Let ρ and r be C^4 respective strictly

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