ASIAN J. MATH.
Vol. 11, No. 1, pp. 167–178, March 2007

013

ON THE CONTINUITY PRINCIPLE*

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To Salah Baouendi on the occasion of his seventieth birthday

Key words. Continuity principle, envelopes of holomorphy, polynomial hulls

AMS subject classifications. Primary 32D10; Secondary 32E20, 32D20, 32D26

The name continuity principle is related to one of the first basic observations in several complex variables. It goes back to the work of Hartogs and Cartan–Thullen. This principle provides us the classical tool for obtaining compulsory analytic continuation in the following sense. There are domains in $\mathbb{C}^n$, $n > 1$, with the property that all analytic functions in the domain have analytic extension to a larger domain. The following classical form of the continuity principle uses continuous families of analytic discs. We will formulate it only for the case of complex dimension $n = 2$ and we will restrict ourselves to this case throughout the paper.

Denote by $c \subset \mathbb{C}^2$ the set $c = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, z_2 = 0\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = 1, z_2 \in [0, 1]\}$ and by C its convex hull, $C = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, z_2 \in [0, 1]\}$. Note that c is topologically a half-sphere with boundary a circle (equivalently, a closed disc; for avoiding confusion with analytic discs we prefer to speak about half-spheres). For a small positive number ϵ we denote by c_ϵ, respectively,. C_ϵ, the corresponding ϵ-neighbourhoods. Let $F : C_\epsilon \to \mathbb{C}^2$ be a locally biholomorphic mapping. The continuity principle says that any analytic function f in $F(c_\epsilon)$ has analytic continuation to a neighbourhood of any point in $F(C_\epsilon)$. (But in general, there is no analytic function in $F(C_\epsilon)$ which is an extension of the original function f).

The advantage of the continuity principle is that it is geometric in its nature. It is well-known that the envelope of holomorphy of a domain D in $\mathbb{C}^n$ (the "largest" Riemann domain over $\mathbb{C}^n$ to which all analytic functions in D have analytic extension) can be obtained by a successive procedure, consisting in gluing to the preceding set families of immersed analytic discs (Levi-flat 3-balls) $F(C)$ along half-spheres $F(c)$. Even this simple approach allows to make some statements about the envelope of holomorphy of some domains.

For further applications, in particular for making guesses concerning envelopes of holomorphy in more subtle situations, more general and flexible versions of the continuity principle are helpful. It is not easy to find such more general versions and mistakes have been made in the literature. One would like to replace families of immersed analytic discs by families of Riemann surfaces with boundary (bordered Riemann surfaces for short), or, more generally, by families of one-dimensional analytic varieties with boundary or by 1-chains, provided a suitable condition of semi-continuity of these objects and their boundaries is given.

Here we will state the version of continuity principle given in [JP].

Start with the following observation (which we explain just for the sake of simplicity only in case the mapping F is injective on C_ϵ). Namely, the aforementioned

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