ASIAN J. MATH.
Vol. 11, No. 1, pp. 151–166, March 2007

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MANIFOLDS ADMITTING GENERIC IMMERSIONS INTO $\mathbb{C}^N$*

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Dedicated to Salah Baouendi in friendship and respect on the occassion of his 70^{th} birthday

Key words. Generic immersion, totally real immersion, Gromov theory

AMS subject classifications. 57R42, 32Q99

The purpose of this paper is to examine bundle-theoretic conditions which are equivalent to a manifold admitting a generic immersion into $\mathbb{C}^N$. In the first section we state the main result and discuss its application to the most familiar case of totally real immersions. We establish the main result in the second section, by applying Gromov theory as presented by Eliashberg and Mishachev [7]. We discuss a technical point related to complex structures in section 3, which shows the limitations of the argument in the previous section. Section 4 is a summary of results, mostly established in the 1980's, on totally real immersions and embeddings; these are presented from a topological perspective. Open questions are collected in the final section.

1. Generic Immersions. Let n and k be integers with $n \geq 0$ and $k \geq 1$. Call an immersion $\pi : M^{2n+k} \to \mathbb{C}^{n+k}$ *generic* if it satisfies any of the following equivalent conditions (here J denotes the standard complex structure on the tangent bundle to $\mathbb{C}^{n+k}$):

1. At each point $p \in M$, the real vector space $\pi_*TM \cap J\pi_*TM$ has the smallest possible dimension (which is $2n$).
2. At each point $p \in M$, $\pi_*TM + J\pi_*TM = T\mathbb{C}^{n+k}|_{\pi(p)}$.
3. At each point $p \in M$, the complex vector space $\pi_*\mathbb{C}TM \cap T^{0,1}(\mathbb{C}^{n+k})$ has the smallest possible dimension (which is n).
4. At each point $p \in M$, $\pi^*(dz_1 \wedge \cdots \wedge dz_{n+k}) \neq 0$ where $(z_1, \ldots, z_{n+k})$ are the usual coordinates on $\mathbb{C}^{n+k}$.
5. At each point $p \in M$, the map $\Psi : \mathbb{C}T_pM \to T^{0,1}_{\pi(p)}(\mathbb{C}^{n+k})$ is surjective, where Ψ is given by projecting
$$\pi_*\zeta \in \mathbb{C}T_{\pi(p)}\mathbb{C}^{n+k} = T^{1,0}_{\pi(p)}(\mathbb{C}^{n+k}) \oplus T^{0,1}_{\pi(p)}(\mathbb{C}^{n+k})$$
into the second factor.

   And for generic *embeddings*, we add one more equivalent condition:

6. $\pi(M)$ is given by equations
$$\rho_j(z, \bar{z}) = 0, \quad j = 1, \ldots, k$$
on $\mathbb{C}^{n+k}$ with
$$\partial\rho_1 \wedge \cdots \wedge \partial\rho_k \neq 0 \text{ at each point of } \pi(M).$$

*Received August 31, 2006; accepted for publication February 9, 2007.
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