ASIAN J. MATH.
Vol. 11, No. 1, pp. 141–150, March 2007

011

NON-UNIFORM CONTINUITY IN H^1 OF THE SOLUTION MAP OF THE CH EQUATION*

A. ALEXANDROU HIMONAS†, GERARD MISIOŁEK‡, AND GUSTAVO PONCE§

Abstract. We show that the solution map of the Camassa-Holm equation is not uniformly continuous in the initial data in the Sobolev space of order one on the torus and the real line. The proof relies on a construction of non-smooth travelling wave solutions. We also extend to all H^s an earlier result known to hold for peakons.

Key words. Solution map, uniform contінity, Camassa-Holm equation, travelling waves, Sobolev spaces

AMS subject classifications. Primary: 35Q53

1. Introduction and statement of the result. We study the Cauchy problem for the nonlinearly dispersive Camassa-Holm equation

$$\begin{aligned} &\partial_t u + u\partial_x u + (1-\partial_x^2)^{-1}\partial_x\big(u^2 + \frac{1}{2}(\partial_x u)^2\big) = 0, \\ &u(0) = u_0, \qquad t \geq 0, \;\; x \in \mathbb{T} \text{ or } \mathbb{R}. \end{aligned} \tag{1.1}$$

This equation appeared initially in the context of hereditary symmetries studied by Fuchssteiner and Fokas [FF]. However, it was first written explicitly as a water wave equation by Camassa and Holm [CH], who also studied its "peakon" solutions (see formula (1.2)).

In order to put our work in context it will be helpful to summarize the relevant known results concerning local well-posedness of this equation. In the periodic case the Cauchy problem (1.1) is locally well-posed in the Sobolev space $H^s(\mathbb{T})$ if $s > 3/2$ (see for example [HM1], Danchin [D] or [Mi]), while if $1 \leq s \leq 3/2$ then it is locally well-posed in $H^s(\mathbb{T}) \cap \mathrm{Lip}(\mathbb{T})$ (see DeLellis, Kappeler and Topalov [DKT]) and the solution u depends continuously on initial data u_0 in the H^s-norm. Furthermore, it is also known that the problem (1.1) is locally well-posed in $C^1(\mathbb{T})$ with solutions depending continuously on the data in the C^1-norm (see [Mi]).

Similarly, if $s > 3/2$ then the non-periodic Cauchy problem (1.1) is locally well-posed in $H^s(\mathbb{R})$ with solutions depending continuously on initial data (see Constantin and Escher [CoE], Li and Olver [LO], Rodriguez-Blanco [R], [D] or a survey in Molinet [Mo]).

On the other hand, it was recently shown in [HM3] that for $s \geq 2$ the data-to-solution map $u_0 \to u$ of (1.1) is not uniformly continuous from any bounded set in $H^s(\mathbb{T})$ into $C([0,T], H^s(\mathbb{T}))$. Therefore, in this Sobolev range continuous dependence on the data is the best one can expect. A key step in the proof of that result was a construction of a sequence of smooth travelling wave solutions of the form $u(x,t) = f(x-t)$ depending on two parameters ε and δ, which were related to the maximum

*Received August 22, 2006; accepted for publication March 9, 2007.
†Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA (himonas.1@nd.edu).
‡Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA (gmisiole@nd.edu).
§Department of Mathematics, University of California, Santa Barbara, CA 93106, USA (ponce@math.ucsb.edu).