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THE STRONG OKA'S LEMMA, BOUNDED PLURISUBHARMONIC FUNCTIONS AND THE $\bar{\partial}$-NEUMANN PROBLEM*

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The classical Oka's Lemma states that if Ω is a pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, then $-\log \delta$ is plurisubharmonic where δ is some distance function to the boundary. Let M be a complex hermitian manifold with the metric form ω. Let Ω be relatively compact pseudoconvex domain in M. We say that a distance function δ to the boundary $b\Omega$ satisfies the strong Oka condition if it can be extended from a neighborhood of $b\Omega$ to Ω such that δ satisfies

$$i\partial\bar{\partial}(-\log \delta) \geq c_0\omega \qquad \text{in } \Omega \tag{0.1}$$

for some constant $c_0 > 0$.

In this note we first study the relation between the strong Oka's Lemma and the existence of bounded strictly plurisubharmonic functions on a pseudoconvex domain in a complex manifold with C^2 boundary. The existence of Hölder continuous bounded plurisubharmonic exhaustion functions for pseudoconvex domains with C^2 boundary in $\mathbb{C}^n$ or a Stein manifold is proved by Diederich-Fornaess [DF]. A similar result for domains in $\mathbb{C}P^n$ is proved in Ohsawa-Sibony [OS]. In this paper we will give a more unified approach to the existence of bounded plurisubharmonic functions using the strong Oka's lemma. We also show the existence and regularity of the $\bar{\partial}$-Neumann operator for pseudoconvex domains in complex Kähler manifolds with nonnegative curvature, extending earlier results of [BC] and [CSW]. There are also many other applications of bounded plurisubharmonic functions (see [DF], [GW] and [De]).

A quantitative approach to the strong Oka's Lemma with compactness and subellipticity for the $\bar{\partial}$-Neumann operator has been obtained recently in [Ha] for pseudoconvex domains in $\mathbb{C}^n$. In particular, if (0.1) is satisfied for some positive continuous function $c_0 = c_0(z)$ for $z \in \Omega$ and $c_0(z) \to \infty$ as $z \to b\Omega$, then the $\bar{\partial}$-Neumann operator is compact. Furthermore, the $\bar{\partial}$-Neumann operator is subelliptic if c_0 in (0.1) goes to infinity by some inverse fractional order of the distance function. Such results hold for a pseudoconvex domain with Lipschitz boundary in $\mathbb{C}^n$. This gives another description of the finite type condition for pseudoconvex domains (see Kohn [Ko1], D'Angelo [DA] and Catlin [Ca]). In Section 3 we give a more streamlined proof of this result when the domain is C^2 pseudoconvex in a complex Kähler manifold with nonnegative curvature. All our results are stated for domains with C^2 boundary. It

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