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THE X-VARIETIES FOR CR MAPPINGS BETWEEN HYPERQUADRICS*

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Dedicated to Salah Baouendi as his seventieth birthday approaches

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0. Introduction. Let M and M' denote smooth CR submanifolds of complex Euclidean spaces of possibly different dimensions. One of the most basic problems in CR geometry is to relate properties of the collection of smooth CR mappings $f : M \to M'$ to the geometries of M and M'. See the monograph [BER] for the complete theory of CR mappings and references.

Perhaps the simplest question about CR mappings is whether, given M and M', there exist any nonconstant smooth CR mappings between M and M'. For spheres and hyperquadrics there are many examples; as the target dimension increases the dimension of the moduli space of examples is unbounded above. A CR mapping f is a solution of a first order system of partial differential equations, but the information that $f(M) \subset M$ is a nonlinear condition on f. When M and M' are spheres or hyperquadrics, the approach in this paper eliminates the nonlinear aspect of the problem.

This paper considers CR mappings between hyperquadrics, including spheres as a special case. We are interested in the structure of CR mappings from a given hyperquadric to hyperquadrics in all possible target dimensions. Placing restrictions on the dimension of the target and on the signature then places restrictions on the possible mappings. In Section 1 we study a certain variety X_f naturally associated with a holomorphic mapping f such that $f(M) \subset M'$, and we use it to partially address the issue of the complexity of rational mappings taking M to M'. Homogenization plays a crucial role in our study; the variety X_f is easier to understand when f is homogeneous. In Section 2 we further analyze the homogeneous case, generalizing work done by the author for spheres [D2], [D3].

The complexity question is interesting and quite difficult even when M and M' are spheres. Write $||z||^2$ for the squared Euclidean norm in all dimensions. Let M denote the unit sphere given by $\{z : ||z||^2 = 1\}$ in $\mathbf{C}^n$ and M' the unit sphere in $\mathbf{C}^N$. For $n \geq 2$, Forsteneric [F1] proved that a smooth (infinitely differentiable) CR mapping between spheres must be the restriction of a rational mapping to the sphere. See also [CS]. The degree of a rational mapping provides one measure of its complexity.

When $1 = n \leq N$, there is no restriction on the degree of a rational mapping sending the circle to a sphere. For $n \geq 2$, Forstneric [F1] also found a crude bound on the degree of the rational mapping (from sphere to sphere) in terms of the dimensions

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