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PERIODIC SOLUTIONS FOR A FAMILY OF EULER-LAGRANGE SYSTEMS*

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Dedicated to Professor Salah Baouendi on his seventieth birthday

Abstract. In this article we study the geometry induced by the sub-Laplacian $X_1^2 + X_2^2$ with $X_1 = \partial_{x_1} + A_1(x)\partial_t$ and $X_2 = \partial_{x_2} - A_2(x)\partial_t$. Here A_1, A_2 are two smooth functions defined on $\mathbb{R}^3$ such that $\varphi(x) := \frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1} \neq 0$. We first characterize necessary and sufficient conditions for horizontal curves. Then we solve the Euler-Lagrange system explicitly when φ is linear. Moreover, we show that the solutions for the system is periodic and the Lagrange multipliers depend on the 1-connection form $\omega = dt - A_1(x)dx_1 + A_2(x)dx_2$. Therefore, the arc lengths of geodesics can be computed explicitly. We also study abnormal minimizers in the last section.

Key words. subRiemannian geometry, horizontal curves, elliptic functions, Euler-Lagrange system, Jacobi's epsilon function

AMS subject classifications. Principal 53C17; Secondary 34K10, 35H20

1. Introduction and Background. Let $X_1, \dots, X_m$ be m linearly independent vector fields defined on an n-dimensional real manifold $\mathcal{M}$ with $m \leq n$. To induce a geometry on $\mathcal{M}$ we assume that the set of "horizontal" vector fields, or given directions, $X = \{X_1, \dots, X_m\}$ is an orthonormal set. More precisely, let h be a positive definite inner product defined on $\mathcal{D} = \text{span}\{X_1, \dots, X_m\}$ such that $h(X_j, X_k) = \delta_{jk}$ with $1 \leq j, k \leq m$. If $m = n$, this yields a Riemannian metric on $\mathcal{M}$. If $m < n$, we need further assumptions on the vector fields $\{X_1, \dots, X_m\}$. A subRiemannian manifold $(\mathcal{M}, \mathcal{D}, h)$ is called a Heisenberg manifold if for any 1-form ω with $\ker(\omega) = \mathcal{D}$, one has

$$\det \omega([X_j, X_k])_{jk} \neq 0 \tag{1.1}$$

on $\mathcal{M}$. It is easy to see that the distribution $\mathcal{D}$ is not integrable because otherwise it would be involutive and hence $\omega([X_j, X_k]) = 0$. Hence, the equation (1.1) won't hold. The vectors $V \in \mathcal{D}_p$ are called *horizontal vectors* at $p \in \mathcal{M}$. The distribution $\mathcal{D}$ is also called *horizontal distribution.* The sections of the *horizontal bundle* $\mathcal{D}$ are called vector fields $X \in \Gamma(\mathcal{D})$. In other words, X is a smooth assignment $\mathcal{M} \ni p \to X_p \in \mathcal{D}_p$. A curve $\gamma : [0, \tau] \to \mathcal{M}$ is called a *horizontal curve* if $\dot\gamma(s) \in \mathcal{D}_{\gamma(s)}$ for all $s \in [0, \tau]$, *i.e.,*

$$\dot\gamma(s) = \sum_{j=1}^{m} a_j(s) X_j.$$

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