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CAUCHY INTEGRALS AND MÖBIUS GEOMETRY OF CURVES*

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Define a line bundle E over the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by gluing together two copies of $\mathbb{C} \times \mathbb{C}$ using the transition mapping $(z, w) \mapsto (1/z, iw/z)$. Since the square of this bundle is the holomorphic cotangent bundle of $\widehat{\mathbb{C}}$ it is reasonable to use the notation $f(z)\sqrt{dz}$ for sections of E.

If $f(z)$ is holomorphic on $R < |z| < \infty$ then $f(z)\sqrt{dz}$ extends to a holomorphic section of E over $R < |z| \le \infty$ if and only if $f(z) \to 0$ as $z \to \infty$.

A Möbius transformation $T : z \mapsto \frac{az+b}{cz+d}$ of $\widehat{\mathbb{C}}$ lifts to (a pair of) maps $E \to E$ given by $(z, w) \mapsto \left(Tz, \frac{\sqrt{ad-bc}}{cz+d} w\right)$. Note that the norm

$$\|F\|_\gamma = \sqrt{\int_\gamma |f(z)|^2 \, |dz|} \tag{1}$$

of a section $F = f(z)\sqrt{dz}$ of E over a curve γ in $\widehat{\mathbb{C}}$ is Möbius-invariant.

Given a smooth Jordan curve $\gamma \subset \mathbb{C}$ and a section $F = f(z)\sqrt{dz}$ of E over γ then the Cauchy integral

$$\frac{1}{2\pi i} \int_{w\in\gamma} f(w)\sqrt{dw}\, \frac{\sqrt{dz}\sqrt{dw}}{w-z} \tag{2}$$

defines a pair of holomorphic sections $\mathcal{C}^{\text{in}}_\gamma F$ and $\mathcal{C}^{\text{out}}_\gamma F$ over the two components γ^{in} and γ^{out} of $\widehat{\mathbb{C}} \setminus \gamma$ with the property that F is the jump

$$\text{b. v. } \mathcal{C}^{\text{in}}_\gamma F - \text{b. v. } \mathcal{C}^{\text{out}}_\gamma F$$

between boundary values of $\mathcal{C}^{\text{in}}_\gamma F$ and $\mathcal{C}^{\text{out}}_\gamma F$; moreover, the operators $\mathcal{C}^{\text{in}}_\gamma$ and $\mathcal{C}^{\text{out}}_\gamma$ are completely characterized by this description. (This is a restatement of the classical Plemelj formula – see for example [Hen, §14.1].)

Since the characterization of $\mathcal{C}^{\text{in}}_\gamma F$ and $\mathcal{C}^{\text{out}}_\gamma F$ given above is Möbius-invariant it follows that the operators $\mathcal{C}^{\text{in}}_\gamma$ and $\mathcal{C}^{\text{out}}_\gamma$ are Möbius-invariant. (This can also be checked by a direct computation verifying that the Cauchy kernel $\frac{1}{2\pi i}\frac{\sqrt{dz}\sqrt{dw}}{w-z}$ is Möbius-invariant.)

We also consider the following Möbius-invariant operators.

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