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THE EULER CHARACTERISTIC AND FINITENESS OBSTRUCTION OF MANIFOLDS WITH PERIODIC ENDS*

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Abstract. Let M be a complete orientable manifold of bounded geometry. Suppose that M has finitely many ends, each having a neighborhood quasi-isometric to a neighborhood of an end of an infinite cyclic covering of a compact manifold. We consider a class of exponentially weighted inner products $(\cdot,\cdot)_k$ on forms, indexed by $k > 0$. Let δ_k be the formal adjoint of d for $(\cdot,\cdot)_k$. It is shown that if M has finitely generated rational homology, $d+\delta_k$ is Fredholm on the weighted spaces for all sufficiently large k. The index of its restriction to even forms is the Euler characteristic of M.

This result is generalized as follows. Let $\pi = \pi_1(M)$. Take $d+\delta_k$ with coefficients in the canonical $C^*(\pi)$-bundle ψ over M. If the chains of M with coefficients in ψ are $C^*(\pi)$-finitely dominated, then $d+\delta_k$ is Fredholm in the sense of Miščenko and Fomenko for all sufficiently large k. The index in $\tilde{K}_0(C^*(\pi))$ is related to Wall's finiteness obstruction. Examples are given where it is nonzero.

Key words. Index theory, complete manifold, weighted cohomology, Euler characteristic, Wall obstruction, K-theory.

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0. Introduction. The analytic index of the operator $d+\delta$ on a compact orientable Riemannian manifold M^n is the Euler characteristic of M, $\chi(M)$. This paper extends this result to a class of complete noncompact manifolds, those with finitely generated rational homology and finitely many quasi-periodic ends. The latter term means that there is a neighborhood of each end which is quasi-isometric to a neighborhood of an end of an infinite cyclic covering of a smooth compact manifold. One reason for interest in such manifolds is a result stated by Siebenmann [34] and proved by Hughes and Ranicki [11]: if M is a manifold of dimension greater than 5 with finitely many ends satisfying a certain tameness condition, then each end has a neighborhood homeomorphic to a neighborhood of an end of an infinite cyclic covering of a compact topological manifold.

$d+\delta$ acting on $\mathcal{L}^2$ forms is a Fredholm operator only in special circumstances. We consider more generally weighted $\mathcal{L}^2$ spaces. These were first used in index theory on manifolds with asymptotically cylindrical ends by Lockhart and McOwen [19] and Melrose and Mendoza. Let $\rho(x)$ be a smooth nonnegative function on M with bounded gradient which tends to ∞ at ∞. Let $k > 0$. The weighted inner product on compactly supported smooth forms is $(u,v)_k = (k^{\rho(x)}u, k^{\rho(x)}v)$, where $(\cdot,\cdot)$ is the $\mathcal{L}^2$ inner product. The weighted forms are obtained by completion. In other words, they are the $\mathcal{L}^2$ space of the measure $k^{2\rho(x)}dx$, where dx is the Riemannian measure. In the quasi-periodic case $\rho(x)$ is chosen to change approximately linearly under iterated covering translations. We consider the operator $D_k = d+\delta_k$, where δ_k is the formal adjoint of d for the weighted inner product. D_k is essentially self-adjoint. We denote by $\bar{D}_k$ the closure of D_k. Let $\bar{D}_k^{even}$ be its restriction to even forms. Let χ and $\chi^{\ell f}$ be the Euler characteristic of the homology and locally finite homology of M. The first main result follows.

THEOREM 0.1. *Let M^n be a complete connected Riemannian manifold of bounded geometry. $\bar{D}_k$ is Fredholm if and only if $\bar{D}_{1/k}$ is, and the indexes satisfy Ind $\bar{D}_{1/k}^{even}$ =*

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