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SYMMETRIES OF ESCHENBURG SPACES AND THE CHERN PROBLEM*

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Dedicated to the memory of S. S. Chern.

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To advance our basic knowledge of manifolds with positive (sectional) curvature it is essential to search for new examples, and to get a deeper understanding of the known ones. Although any positively curved manifold can be perturbed so as to have trivial isometry group, it is natural to look for, and understand the most symmetric ones, as in the case of homogeneous spaces. In addition to the *compact rank one symmetric spaces*, the complete list (see [BB]) of simply connected homogeneous manifolds of positive curvature consists of the *Berger* spaces B^7 and B^{13} [Be], the *Wallach* spaces W^6, W^{12} and W^{24} [Wa], and the infinite class of so-called *Aloff–Wallach* spaces, $\mathcal{A}^7$ [AW]. Their full isometry groups were determined in [Sh2], and this knowledge provided new basic information about possible fundamental groups of positively curved manifolds, and in particular to counter-examples of the so-called *Chern conjecture* (see [Sh1] and [GSh, Ba2]), which states that every abelian subgroup of the fundamental group is cyclic.

Our purpose here is to begin a systematic analysis of the isometry groups of the remaining known manifolds of positive curvature, i.e., of the so-called *Eschenburg spaces*, $\mathcal{E}^7$ [Es1, Es2] (plus one in dimension 6) and the *Bazaikin spaces*, $\mathcal{B}^{13}$ [Ba1], with an emphasis on the former. In particular, we completely determine the identity component of the isometry group of any positively curved Eschenburg space. A member of $\mathcal{E}$ is a so-called *bi-quotient* of SU(3) by a circle:

$$E = \operatorname{diag}(z^{k_1}, z^{k_2}, z^{k_3})\backslash \operatorname{SU}(3)/\operatorname{diag}(z^{l_1}, z^{l_2}, z^{l_3})^{-1},\ |z| = 1$$

with $\sum k_i = \sum l_i$. Further conditions on the integers are required for E to be a manifold and for the Eschenburg metric to have positive curvature, see (1.1). They contain the homogeneous Aloff–Wallach spaces $\mathcal{A}$, corresponding to $l_i = 0, i = 1, 2, 3$, as a special subfamily. Similarly, any member of $\mathcal{B}$ is a bi-quotient of SU(5) by $\operatorname{Sp}(2)\,\mathrm{S}^1$ and the Berger space, $B^{13} \in \mathcal{B}$. It was already noticed several years ago by the first and last author, that both $\mathcal{E}$ and $\mathcal{B}$ contain an infinite family $\mathcal{E}_1$ respectively $\mathcal{B}_1$ of cohomogeneity one, i.e., they admit an isometric group action with 1-dimensional orbit space (see section 1 and [Zi]). There is a larger interesting subclass $\mathcal{E}_2 \subset \mathcal{E}$, corresponding to $l_1 = l_2 = 0$, which contains $\mathcal{E}_1$ as well as $\mathcal{A}$, and whose members

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