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STRUCTURE OF THE TENSOR PRODUCT SEMIGROUP*

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To the memory of S. S. Chern

Abstract. We study the structure of the semigroup $Tens(G)$ consisting of triples of dominant weights (λ, μ, ν) of a complex reductive Lie group G such that

$$(V_\lambda \otimes V_\mu \otimes V_\nu)^G \neq 0.$$

We prove two general structural results for $Tens(G)$ and compute $Tens(G)$ for $G = Sp(4, \mathbb{C})$ and $G = G_2$.

Key words. tensor products; irreducible representations

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1. Introduction. Suppose that G is a complex reductive Lie group. Finite-dimensional irreducible representations V_λ of G are parameterized by their highest weights $\lambda \in \Delta \cap L$, where Δ is the positive Weyl chamber and L is the character lattice of a maximal (split) torus in G. One of the basic questions of the representation theory is to decompose tensor products $V_\lambda \otimes V_\mu$ into sums of irreducible representations. Accordingly, we define the set

$$Tens(G) := \{(\lambda, \mu, \nu) \in (\Delta \cap L)^3 : (V_\lambda \otimes V_\mu \otimes V_\nu)^G \neq 0\}.$$

For a simply-connected Lie group G with root system R we will write $Tens(R)$ instead of $Tens(G)$. It has been known for a long time, see for example [12, Theorem 9.8], that the set $Tens(G)$ forms a semigroup with respect to addition. The goal of this paper is to provide more specific structural theorems for $Tens(G)$ and to compute $Tens(Sp(4, \mathbb{C}))$ and $Tens(G_2)$.

THEOREM 1.1. *For each complex reductive Lie group G, the set $Tens(G)$ is a finite union of elementary subsets of L^3.*

Here an *elementary subset* is a subset given by a finite system of linear inequalities (with integer coefficients) and congruences. Thus, to "describe" $Tens(G)$ one would have to find these inequalities and congruences. The above theorem is an analogue of a theorem of C. Laskowski [17], who proved a similar statement for the structure constants of spherical Hecke rings.

Our next theorem provides a glimpse of what these inequalities and congruences might look like. In [2] and [10] the authors defined a finite-sided homogeneous polyhedral cone $\mathcal{P}(G) = D_3(G/K) \subset \Delta^3$ (where K is a maximal compact subgroup of G), given by the inequalities of the form:

$$\langle \varpi_i, w_1\lambda \rangle + \langle \varpi_i, w_2\mu \rangle + \langle \varpi_i, w_3\nu \rangle \leq 0,$$

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