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MIRROR CONGRUENCE FOR RATIONAL POINTS ON CALABI-YAU VARIETIES*

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0. Introduction. One of the basic problems in arithmetic mirror symmetry is to compare the number of rational points on a mirror pair of Calabi-Yau varieties. At present, no general algebraic geometric definition is known for a mirror pair. But an important class of mirror pairs comes from certain quotient construction. In this paper, we study the congruence relation for the number of rational points on a quotient mirror pair of varieties over finite fields. Our main result is the following theorem:

THEOREM 0.1. *Let X_0 be a smooth projective variety over the finite field $\mathbf{F}_q$ with q elements of characteristic p. Suppose X_0 has a smooth projective lifting X over the Witt ring $W = W(\mathbf{F}_q)$ such that the W-modules $H^r(X, \Omega^s_{X/W})$ are free. Let G be a finite group of W-automorphisms acting on the right of X. Suppose G acts trivially on $H^i(X, \mathcal{O}_X)$ for all i. Then for any natural number k, we have the congruence*

$$\#X_0(\mathbf{F}_{q^k}) \equiv \#(X_0/G)(\mathbf{F}_{q^k}) \pmod{q^k},$$

where $\#X_0(\mathbf{F}_{q^k})$ (resp. $\#(X_0/G)(\mathbf{F}_{q^k})$) denotes the number of $\mathbf{F}_{q^k}$-rational points of X_0 (resp. X_0/G).

The main application of the above theorem is to Calabi-Yau varieties. This gives the following theorem announced in [W], which was the main motivation of the present paper.

THEOREM 0.2. *Let X_0 be a geometrically connected smooth projective Calabi-Yau variety of dimension n over the finite field $\mathbf{F}_q$ with q elements of characteristic p. Suppose X_0 has a smooth projective lifting X over the Witt ring $W = W(\mathbf{F}_q)$ such that the W-modules $H^r(X, \Omega^s_{X/W})$ are free. Let G be a finite group of W-automorphisms acting on the right of X. Suppose G fixes a non-zero n-form on X. Then for any natural number k, we have the congruence*

$$\#X_0(\mathbf{F}_{q^k}) \equiv \#(X_0/G)(\mathbf{F}_{q^k}) \pmod{q^k}.$$

Proof. If X is a Calabi-Yau scheme over W of dimension n, then $H^i(X, \mathcal{O}_X) = 0$ for $i \neq 0, n$ and G acts trivially on them. If the generic fiber of X is geometrically connected, then G acts trivially on $H^0(X, \mathcal{O}_X)$. By Serre duality, $H^n(X, \mathcal{O}_X)$ is dual to $H^0(X, \Omega^n_{X/W})$. Since X is Calabi-Yau, $\Omega^n_{X/W}$ is a trivial invertible sheaf. In order

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