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A SECOND-ORDER INVARIANT OF THE NOETHER-LEFSCHETZ LOCUS AND TWO APPLICATIONS*

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1. Introduction and statement of results. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d cut out by a polynomial

$$F \in k[X_0, \dots X_3].$$

We will be interested in the following questions. What curves does X contain? Can these curves be classified?

For a generic X of degree $d \geq 4$, this question was answered in the 20's, when the Noether-Lefschetz theorem was proved by Lefschetz.

THEOREM 1 (Lefschetz). *If X is a generic smooth surface of degree $d \geq 4$ in $\mathbb{P}^3$ then for any curve $C \subset X$ there exists a surface Y such that $C = X \cap Y$.*

A curve C which has the property that $C = X \cap Y$ for some surface Y will be said to be a complete intersection in X.

This theorem says essentially that if X is generic then the set of curves contained in X is well understood and is as simple as possible. In this article we will study the distribution of surfaces for which the conclusion of Theorem 1 does not hold — or in other words, surfaces containing curves which are not well understood.

Throughout the rest of this article, we will denote by U_d the space parameterising smooth degree d surfaces in $\mathbb{P}^3$. We define the *Noether-Lefschetz locus*, which we denote by NL_d, as follows:

$$X \in NL_d \Leftrightarrow X \text{contains a curve } C \text{ which is not a complete intersection in } X$$

which, by the Leftschetz $(1,1)$ theorem, can alternatively be written as

$$X \in NL_d \Leftrightarrow H^{1,1}_{\text{prim}}(X, \mathbb{Z}) \neq 0.$$

Theorem 1 says that NL_d is a countable union of proper subvarieties of U_d. Throughout the rest of the article, NL will denote one of these subvarieties. Ciliberto et al. showed in [3] that NL_d is dense in the Zariski and complex topologies.

It is interesting to have an idea of the size of the components of NL_d, since this gives us some idea of how rare badly-behaved curves are. An initial (very rough) estimate comes out of Hodge theory. Any component NL can be expressed as the

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