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SUPERSYMMETRIES IN CALABI-YAU GEOMETRY*

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Abstract. We introduce a class of Lie superalgebras, called $\mathfrak{spin}_c$ supersymmetry algebras, constructed from spinor representations. The construction is motivated by supersymmetry algebras used by physicists. On a Riemannian manifold, a Kähler manifold, and a hyperkähler manifold respectively, it is known that some natural operators on the space of differential forms generate certain Lie superalgebras. It turns out that they correspond to $\mathfrak{spin}_c(2)$, $\mathfrak{spin}_c(3)$, and $\mathfrak{spin}_c(5)$ supersymmetry algebras respectively. Motivated by Mirror Symmetry Conjecture, we also consider supersymmetries on Calabi-Yau manifolds.

Key words. supersymmetry, Calabi-Yau manifolds

AMS subject classifications. 53Z05

The supersymmetry (SUSY) algebra [10] is a special kind of Lie superalgebras [6] which involves spinor representations. It was invented by physicists in the seventies to formulate a unified theory for fermions and bosons. A guiding principle in physics is to examine the symmetries of Lagrangians. Quite often a classical Lagrangian can be extended to have supersymmetries. For example, Donaldson theory has been interpreted by Witten [12] as a twisted $N = 2$ supersymmetric quantum field theory which extends the Lagrangian of the classical Yang-Mills theory. The study of this theory has led to Seiberg-Witten theory [13]. Other examples include supersymmetric extensions of nonlinear sigma models. When the source manifold is a Riemann surface, it turns out [1] that an $N = 1$ supersymmetric extension is always possible; when the target Riemannian manifold is Kähler, an $N = 2$ supersymmetric extension is possible; when the target manifold is hyperkähler, an $N = 4$ supersymmetric extension is possible.

It is well-known to physicists that when one considers the large volume limit, the topological sigma model leads to the space of differential forms on the target manifold and differential operators on it. So it is conceivable that supersymmetries in the topological sigma model leads to some supersymmetries among these operators. One is naturally led to the problem of finding a relationship between manifolds with special holonomy groups and the supersymmetry algebras formed by differential operators on them. Another motivation for this problem is to find the analogues of Kähler manifolds etc. in non-commutative geometry. This is discussed in a recent paper by Fröhlich, Grandjean and Recknagel [3]. Their idea is as follows: since the space of exterior forms is a super vector space, the space of linear operators on it is naturally a Lie superalgebra (actually a Poisson superalgebra) under the supercommutators. For manifolds with special holonomy groups, one can find operators which generate finite dimensional Lie (super)algebras.

Actually some examples of this type have been well-known. In Riemannian geometry, Witten considered the following very simple Lie superalgebra in his influential paper on Morse theory [11]: for a Riemannian manifold (M, g), let $d : \Omega^*(M) \to \Omega^*(M)$

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