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CRITICAL SOBOLEV EXPONENT PROBLEM IN A BALL WITH NONLINEAR PERTURBATION CHANGING SIGN*

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1. Introduction. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain with smooth boundary. Consider the following problem:

$$\begin{aligned} -\Delta u &= u^{\frac{n+2}{n-2}} + \rho(u) \quad && \text{in } \Omega\ , \\ u &> 0 && \text{in } \Omega\ , \\ u &= 0 && \text{on } \partial\Omega\ , \end{aligned} \tag{1.1}$$

where $\rho : \overline{\mathbb{R}}^+ \to \mathbb{R}$ is a smooth function with $\rho(0) = 0$. In [3], Brezis and Nirenberg have studied the above problem when $\rho \geq 0$. Among many interesting results they have shown that if $\rho(s) = o(s)$ as $s \to 0, \rho(s) = o(s^{(n+2)/(n-2)})$ as $s \to \infty$ and

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^{\varepsilon^{-1/2}} \tilde{\rho}\big[\big(\frac{\varepsilon^{-1/2}}{1+s^2}\big)^{\frac{n-2}{2}}\big] s^{n-1}\, ds = \infty\ , \tag{1.2}$$

then (1.1) admits a solution, where $\tilde{\rho}(s) = \int_0^s \rho(t)dt$ is the primitive of ρ. In particular, if $n \geq 5$ and $s^{-n/n-2}\tilde{\rho}(s) \to 0$ as $s \to \infty$, then after a change of variables and integration by parts, condition (1.2) is equivalent to the following condition:

$$\int_0^\infty \rho(s) s^{-n/(n-2)}\, ds > 0\ . \tag{1.3}$$

As a consequence, if $n \geq 5, \rho \not\equiv 0, \rho \geq 0$ and has compact support, then (1.3) holds and hence (1.1) admits a solution.

So far the above results are for $\rho \geq 0$. Now the question is what happens if ρ changes sign. In this regard the following result has been proved in [1]. Let $\rho(s) = (s+1)^{(n+2)/(n-2)} - s^{(n+2)/(n-2)} - (s+1)$. Then for $n \geq 7$, ρ changes sign

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