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ON SOME NONLINEAR EQUATIONS INVOLVING THE p-LAPLACIAN WITH CRITICAL SOBOLEV GROWTH

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I. Introduction. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let also $p \in (1, n)$ real, and a, f two smooth functions on $\overline{\Omega}$. We are concerned here with the existence of solutions $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, positive or not, to the problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)|u|^{p-2}u = f(x)|u|^{p^\star-2}u \ \text{ in } \ \Omega \\ u \not\equiv 0\ ,\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{$\star$}$$

where $p^\star = np/(n-p)$. The first term in the LHS of $(\star)$ is called the p-laplacian of u, while changing sign solutions to $(\star)$ are called nodal solutions. When $p = 2$, the p-Laplacian is just the usual laplacian (with the minus sign convention). Equation $(\star)$ is then said to be of scalar curvature type, in reference with the equation relating the scalar curvatures of two conformal Riemannian metrics. Such an equation has been studied by many authors. A remark one has to do here is that since $q = p^\star$ is critical for the embedding of $W^{1,p}$ in L^q, it is not possible to obtain solutions of $(\star)$ via simple variational arguments. Problem $(\star)$ has been studied by Guedda and Veron [8]. Among other results, these authors proved that for $a \equiv 0$, the equation

$$-\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = |u|^{p^\star-1}$$

admits no nonzero solution in $W_0^{1,p}(\Omega)$ if Ω is starshaped with respect to some point, while for a constant negative, greater than minus the best Poincaré constant of $W_0^{1,p}(\Omega)$, and $p^2 \le n$, the problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)u^{p-1} = u^{p^\star-1} \ \text{ in } \ \Omega \\ u > 0 \ \text{ in } \ \Omega\ ,\ u = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

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