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ASYMPTOTIC BEHAVIOUR OF A NONLINEAR ELLIPTIC EQUATION WITH CRITICAL SOBOLEV EXPONENT: THE RADIAL CASE

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1. Introduction and statement of the results

Let B be the unit ball in $\mathbb{R}^N$, $N \geq 3$, and $a, f : \mathbb{R} \to \mathbb{R}$ two smooth functions. We regard $x \mapsto a(|x|)$ and $x \mapsto f(|x|)$ as functions of the variable $x \in \mathbb{R}^N$. As is easily seen, these functions are locally Lipschitz. In particular, they are locally in $C^{0,\alpha}$ for all $\alpha \in (0,1)$. In order to fix ideas, we suppose that $f > 0$ and that $f(0) = 1$. Then we consider the following problem:

$$(I) \begin{cases} \Delta u + a(|x|)u = N(N-2)f(|x|)u^p \text{ in } B \\ u > 0 \text{ in } B, \ \ u = 0 \text{ on } \partial B, \end{cases}$$

where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplacian with the minus sign convention, and $p = \frac{N+2}{N-2}$ is critical from the viewpoint of Sobolev embeddings. We let $H_0^1(B)$ be the standard Sobolev space, defined as the completion of $\mathcal{D}(B)$, the set of smooth functions with compact support in B, with respect to the norm $\|u\| = \sqrt{\int_B |\nabla u|^2\, dx}$. In the sequel, we suppose that the operator $u \mapsto \Delta u + a(|x|)$ is coercive on $H_0^1(B)$. This is the case when $a > -\lambda_1$, where λ_1 is the first eigenvalue of Δ for the Dirichlet problem.

Situations where (I) does not have a solution are in Pohozaev [14]. In particular, (I) does not possess a solution if $a \equiv 0$ and $f \equiv 1$. However, as it is subcritical from the viewpoint of Sobolev embeddings, the problem

$$(I_\epsilon) \begin{cases} \Delta u_\epsilon + a(|x|)u_\epsilon = N(N-2)f(|x|)u_\epsilon^{p-\epsilon} \text{ in } B \\ u_\epsilon > 0 \text{ in } B, \ \ u_\epsilon = 0 \text{ on } \partial B \end{cases}$$

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