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# SOME NEAR CRITICAL PROBLEMS

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In this paper, we consider the exterior problem

$$\begin{aligned} -\Delta u &= u^{p*-\epsilon} \quad \text{in} \quad R^N \backslash \Omega \\ u &> 0 \\ u &= 0 \qquad \text{in} \ \ \partial\Omega \ , \end{aligned} \tag{1}$$

where $\Omega$ is a bounded open set in $R^N$ with smooth boundary ($\Omega$ not necessarily connected), $p_* = (N+2)(N-2)^{-1}, N > 2$ and $R^N \backslash \Omega$ is connected. We prove under a technical condition that, if $\Omega$ has non-trivial reduced homology with coefficients in $Z_2$ and if $\epsilon$ is small and positive, then (1) has a solution. This partially proves a conjecture of the author in [9]. Note that a weaker result was announced in [7]. Note also that counterexamples in [9] show that the result need not be true if $\epsilon$ is not small. (In the counterexamples, $\Omega$ is not connected or $p_* - \epsilon$ is close to but larger than $N/(N-2)$.) The above problem originally arose in [9] as a natural limit problem for superlinear problems in domains with small holes. Note that the solutions we construct do not have sharp peaks and have relatively fast decay at infinity. We do not know any example where our technical condition fails.

The strategy of our proof is to use a Kelvin transformation to obtain a problem on a bounded domain $\Omega^*$ and then consider our problem as a perturbation of the pure (i.e., $\epsilon = 0$) critical exponent problem

$$\begin{aligned} -\Delta u &= u^{p*} \quad \text{in} \ \ \Omega^* \\ u &= 0 \quad \ \text{on} \ \ \partial\Omega^* \ . \end{aligned} \tag{2}$$

It then largely reduces (by [8], §3) to proving that the critical exponent problem has a compact set of solutions which "carries topology". The proof of this requires a careful examination of the Bahri-Coron proof [2] for the critical exponent problem using some ideas in [18]. There is a technical

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