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FLAT SURFACES WITH SINGULARITIES IN EUCLIDEAN 3-SPACE

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Abstract

It is classically known that complete flat (that is, zero Gaussian curvature) surfaces in Euclidean 3-space $\boldsymbol{R}^3$ are cylinders over space curves. This implies that the study of global behaviour of flat surfaces requires the study of singular points as well. If a flat surface f admits singularities but its Gauss map ν is globally defined on the surface and can be smoothly extended across the singular set, f is called a *frontal*. In addition, if the pair (f, ν) defines an immersion into $\boldsymbol{R}^3 \times S^2$, f is called a *front*. A front f is called *flat* if the Gauss map degenerates everywhere. The parallel surfaces and the *caustic* (i.e. focal surface) of a flat front f are also flat fronts. In this paper, we generalize the classical notion of completeness to flat fronts, and give a representation formula for a flat front which has a non-empty compact singular set and whose ends are all immersed and complete. As an application, we show that such a flat front has properly embedded ends if and only if its Gauss map image is a convex curve. Moreover, we show the existence of at least four singular points other than cuspidal edges on such a flat front with embedded ends, which is a variant of the classical four vertex theorem for convex plane curves.

Introduction

In this paper, we shall investigate the global behaviour of flat surfaces with singularities in Euclidean 3-space $\boldsymbol{R}^3$. In fact, for the study of global properties of flat surfaces, considering only immersions is too restrictive, as is already clear from the classical fact (Fact 0.1) below.

Let M^2 be a smooth 2-manifold and $f : M^2 \to \boldsymbol{R}^3$ a C^∞-map. A point $p \in M^2$ is called *regular* if f is an immersion on a sufficiently small neighborhood of p, and is called *singular* if it is not regular. If f is an immersion and has zero Gaussian curvature, it is called a (regular) flat surface. It is classically known that regular flat surfaces have open dense

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