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WILLMORE IMMERSIONS AND LOOP GROUPS

FRÉDÉRIC HÉLEIN

Abstract

We propose a characterisation of Willmore immersions inspired from the works of R. Bryant on Willmore surfaces and J. Dorfmeister, F. Pedit, H.-Y. Wu on harmonic maps between a surface and a compact homogeneous manifold using moving frames and loop groups. We use that formulation in order to construct a Weierstrass type representation of all conformal Willmore immersions in terms of closed one-forms.

Let $\mathbb{R}^3$ be the Euclidean space and let us consider the set $\mathcal{D}$ of all compact, oriented surfaces without boundary which are immersed in $\mathbb{R}^3$ (the immersion being of class $\mathcal{C}^k$ for $k \geq 4$). For a surface $\mathcal{S} \in \mathcal{D}$ we consider the area 2-form dA induced by the first fundamental form of the immersion on $\mathcal{S}$ and the principal curvatures $k_1 \leq k_2$ computed using the first and the second fundamental forms. A point of $\mathcal{S}$ such that $k_1 = k_2$ is called an *umbilic point.* Let $H := (k_1 + k_2)/2$ be the mean curvature and $K := k_1 k_2$ the Gauss curvature. The quantity

$$\mathcal{W}(\mathcal{S}) := \int_{\mathcal{S}} H^2 dA$$

defines a functional on $\mathcal{D}$ called *Willmore functional.* A variant of $\mathcal{W}$ is

$$\tilde{\mathcal{W}}(\mathcal{S}) := \int_{\mathcal{S}} \frac{1}{4}(k_1 - k_2)^2 dA,$$

which differs from $\mathcal{W}(\mathcal{S})$ by $\mathcal{W}(\mathcal{S}) - \tilde{\mathcal{W}}(\mathcal{S}) = \int_{\mathcal{S}} K dA = 4\pi(1-g)$ where g is the genus of $\mathcal{S}$. Both functionals having the same critical points on $\mathcal{D}$ called *Willmore surfaces*; they are solutions of the equation

$$\Delta H + 2H(H^2 - K) = 0,$$

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