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ON A MONGE-AMPÈRE EQUATION ARISING IN GEOMETRIC OPTICS

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Abstract

In this paper we study a Monge-Ampère equation arising in geometric optics. We will establish the a priori estimates and derive the existence of solutions by the continuity method. We also give a Legendre-type transformation for this equation.

1. Introduction

We consider here an equation of Monge-Ampère type which arises in geometric optics. Suppose a point source of light is located at the origin $O \in \mathbb{R}^3$ and let Γ be a closed surface which is star-shaped with respect to the origin. If we identify each direction of the ray with a point on S^2, and the ray of the light reflects according to geometric optics, then the direction of the reflection defines a point on S^2. Hence we obtain a map from S^2 to S^2. In [26], as a part of Problem 21, Yau asked: "How much information does this map tell us about the surface?" Let Γ be represented as a graph over the unit sphere S^2, $\Gamma = \{x \cdot \rho(x);\ \ x \in S^2\}$. Let $\gamma(x)$ denote the unit outer normal of Γ at $x \cdot \rho(x)$, and $y = T(x) = T_\rho(x)$ the direction of the light reflected by Γ. Here we regard a unit vector as a point on S^2. By the reflection law we have

$$y = x - 2\langle x, \gamma\rangle\gamma.$$

Let $f(x)$ denote the intensity of the source O, and $g(y)$ the distribution of the directions of the reflected light on S^2. Both f and g are non-negative and measurable. Suppose no energy is lost in reflection, and

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