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SOLITON EQUATIONS AND DIFFERENTIAL GEOMETRY

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1. Introduction

In this paper we study certain symplectic, Lie theoretic, and differential geometric properties of soliton equations.

The equation for harmonic maps from the Lorentz space $R^{1,1}$ to a symmetric space, and the equation for isometric immersions of space forms into space forms have many of the same properties as soliton equations—for example, they have Lax pairs and Bäcklund transformations—and two of the main goals of this paper are to find Hamiltonian formulations for these equations and to see how they fit into the general theory of soliton equations. As a by-product, we also find many new n-dimensional soliton systems.

It is well-known that most finite-dimensional, completely integrable, Hamiltonian systems can be obtained by applying the Adler-Kostant-Symes Theorem (AKS) to some Lie algebra $\mathcal{G}$ equipped with an ad-invariant, non-degenerate bi-linear form, and a decomposition $\mathcal{G} = \mathcal{K} + \mathcal{N}$. The symplectic manifold is some co-adjoint N-orbit $M \subset \mathcal{K}^\perp \simeq \mathcal{N}^*$ and the equation is the Hamiltonian equation of $f \mid M$, where $f : \mathcal{G} \to R$ is some suitable Ad-invariant function. For example, Kostant obtained the generalized Toda lattice ([23]) by applying the AKS theorem to $\mathcal{G} = \mathcal{K} + \mathcal{N}$ such that the corresponding G/K is a non-compact, symmetric space of split type (i.e., the rank of G/K is equal to the rank of G), and Adler and Van Moerbeke obtained the Euler-Arnold equation and Moser's geodesic flow on the ellipsoid ([6]) by applying the AKS theorem to $\mathcal{G} =$ the loop algebra of a simple Lie algebra.

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