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BUBBLE TREE CONVERGENCE FOR HARMONIC MAPS

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Abstract

Let Σ be a compact Riemann surface. Any sequence $f_n : \Sigma \to M$ of harmonic maps with bounded energy has a "bubble tree limit" consisting of a harmonic map $f_0 : \Sigma \to M$ and a tree of bubbles $f_k : S^2 \to M$. We give a precise construction of this bubble tree and show that the limit preserves energy and homotopy class, and that the images of the f_n converge pointwise. We then give explicit counterexamples showing that bubble tree convergence fails (i) for harmonic maps f_n when the conformal structure of Σ varies with n, and (ii) when the conformal structure is fixed and $\{f_n\}$ is a Palais-Smale sequence for the harmonic map energy.

Consider a sequence of harmonic maps $f_n : \Sigma \to M$ from a compact Riemann surface (Σ, h) to a compact Riemannian manifold (M, g) with bounded energy

$$E(f_n) = \frac{1}{2}\int_\Sigma |df_n|^2 \leq E_0. \tag{0.1}$$

Such a sequence has a well-known "Sacks-Uhlenbeck" limit consisting of a harmonic map $f_0 : \Sigma \to M$ and some "bubbles" — harmonic maps $S^2 \to M$ obtained by a renormalization process. In fact, by following the procedure introduced in [12], one can modify the Sacks-Uhlenbeck renormalization and iterate, obtaining bubbles on bubbles. The set of all bubble maps then forms a "bubble tree" ([12]). One would like to know in precisely what sense the sequence $\{f_n\}$ converges to this bubble tree. The major issue is the appearance of "necks" joining one bubble to the next.

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