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# METRIC STRUCTURE OF CUT LOCI IN SURFACES AND AMBROSE'S PROBLEM

JAMES J. HEBDA

**Abstract**

It is shown that every compact subset of the cut locus of a point in a complete, two-dimensional Riemannian manifold has finite one-dimensional Hausdorff measure. When combined with a result from an earlier paper, this completes the solution of the two-dimensional case of a long-standing problem of Ambrose.

## 1. Introduction

This paper is devoted to the proof of the following theorem.

**Theorem 1.1.** *Let $M$ be a complete, 2-dimensional Riemannian manifold, and let $p$ be a point in $M$. Then every compact subset of the cut locus of $p$ in $M$ has finite 1-dimensional Hausdorff measure.*

Consequently, the cut locus of a point in a complete, two-dimensional Riemannian manifold must satisfy stringent conditions relative to the differentiable structure of the manifold. This means that the problem of determining what subsets of a given surface can be realized as the cut locus of some point for some Riemannian metric depends not only on the topology, but also on the metric structure (or more precisely on the local Lipschitz structure) that they inherit as subsets of the surface. To illustrate this, recall that Gluck and Singer [5] show how a certain subset of the sphere, consisting of countably many great circle arcs radiating out from a common endpoint, can be realized as a cut locus. Altering this set by extending the arcs slightly, so that their lengths form a divergent series, results in a set homeomorphic to the original, but possessing infinite Hausdorff locus. (1-measure, and so no longer realizable as a cut locus (c.f. Example 6.2).

Along the way to proving Theorem 1.1, it is shown that embedded arcs in the cut locus of a point in a surface have finite length (Theorem 4.7). Thus there is an intrinsic metric on the cut locus in which the distance between two cut points is defined to be the length of the shortest arc in

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