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RIGIDITY OF SURFACES WITH NO CONJUGATE POINTS

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Abstract

E. Hopf proved that any complete Riemannian metric with no conjugate points on the torus T^2 is flat. We extend Hopf's argument to obtain sufficient conditions for metrics with no conjugate points on a cylinder or the plane to be flat.

0. Introduction

A complete Riemannian manifold has no conjugate points if any two points in its universal cover are joined by a unique geodesic. The no conjugate point property is a natural generalization of nonpositive curvature: any manifold with nonpositive curvature has no conjugate points by the Cartan-Hadamard theorem. In 1943 E. Hopf proved that a Riemannian metric with no conjugate points on the torus T^2 must be flat [8]. The present paper extends Hopf's arguments to obtain sufficient conditions for metrics on the cylinder $S^1 \times \mathbb{R}$ and the plane $\mathbb{R}^2$ to be flat.

In the case of the cylinder, our main result—Theorem 2.2—is that a cylinder with no conjugate points and curvature bounded from below is flat if its ends do not open out, in other words if there is $L > 0$ such that there is a nontrivial loop of length at most L based at every point. This answers affirmatively a question raised in [6], where the result is proved under the stronger assumption that the cylinder has no focal points. Our method also shows that if the cylinder becomes thin as one approaches both ends, then there must be conjugate points. As a consequence, a cylinder with no conjugate points and curvature bounded from below has infinite area. We do not know whether the lower curvature bound can be removed in these results.

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