J. DIFFERENTIAL GEOMETRY
31 (1990) 601-625

HERMITIAN FINSLER METRICS AND THE KOBAYASHI METRIC

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Abstract

The problem of local equivalence of Hermitian Finsler metrics under holomorphic changes of coordinates is solved. On such a Finsler metric we find some differential conditions which imply that the Finsler metric is the Kobayashi metric of the underlying manifold (these conditions are satisfied if the metric is the Kobayashi metric on a bounded, strictly convex domain in $\mathbb{C}^n$ with smooth boundary).

0. Introduction

The infinitesimal Kobayashi metric is a real-valued function F_M on the tangent bundle of a complex manifold M. For $p \in M$ and $v \in T_pM$,

$$F_M(p, v) = \inf\{1/r\colon \text{there is a holomorphic } f\colon \Delta_r \to M \text{ with } f(0) = p,\ f'(0) - v\},$$

where $\Delta_r = \{z \in \mathbb{C}\colon |z| < r\}$. F_M is clearly an invariant of the complex structure on M, and, indeed, information about F_M can yield information about the complex function theoretic aspects of M (see, e.g., [3], [4]). One can think of $F_M(p, v)$ as being the length of the tangent vector v. One can then define the length of a curve by integrating the length of the tangent vector to the curve, and define a metric d_M by considering the infimum of the lengths of all curves joining two points.

It is natural to try to understand the geometry of this metric. However, the techniques of differential geometry can, in general, be applied only indirectly, because F_M need not have any smoothness, even away from the zero section of the tangent bundle. (For example, the Kobayashi metric on the polydisk in $\mathbb{C}^n$ is not C^1.) However, Lempert [5] has shown that when the complex manifold is a bounded domain $D \subset \mathbb{C}^n$ with smooth, strictly convex boundary, then the Kobayashi metric is smooth. (By smooth we shall always mean C^∞. The results of this paper hold when considering less generous regularity assumptions, but, for what needs to be done here,

Received December 21,1987.