J. DIFFERENTIAL GEOMETRY
30 (1989) 539-554

SCHWARZ'S LEMMA FOR CIRCLE PACKINGS. II

BURT RODIN

To Kotaro Oikawa on his 60th birthday

Introduction

In [6] it was shown that the Riemann mapping function is the limit of circle packing isomorphisms as the circles shrink to points. In [5] and the present paper this connection between conformal mapping and circle packing is investigated further. We are able to estimate the rate of convergence of the circle packing isomorphisms and to obtain information about the relationship of the derivative of the Riemann mapping function to the ratio of the radii of image circle to preimage circle under these isomorphisms.

In [6] the following question was raised: Under the circle packing isomorphisms, does the ratio of the radius of an image circle to that of its preimage converge to the modulus of the derivative of the Riemann mapping function? It is known that the answer is affirmative if the convergence is taken in the L^p norm on compact subsets for some $p > 2$ (see §1). In §4 we prove that the convergence actually holds in the norm of Bounded Mean Oscillation (BMO) on compact subsets (Theorem 6). A key ingredient of the proof is the Schwarz Lemma analog for circle packings [5]. As another application of this Schwarz Lemma analog—specifically, the BMO estimates—we obtain a sufficient condition for the convergence to be uniform. This condition, which is much weaker than the previously known condition, refers to the hexagonal packing constants s_n introduced in [6].

For $n \geq 2$ the constant s_n is defined as the smallest number with the property that, for any circle packing which is combinatorially equivalent to n generations of the regular hexagonal circle packing, the ratio ρ of the radius of a circle of generation 1 to the radius of the circle of generation 0 satisfies $|1 - \rho| \leq s_n$. It is shown in [6] that $s_n \to 0$ as $n \to \infty$.

It had been known previously (see §5) that if $s_n = O(1/n)$ as $n \to \infty$, then the convergence of the ratio-of-radii function to the modulus of the derivative of the Riemann mapping function is uniform on compact

Received July 30, 1987 and, in revised form, April 7, 1988. Research supported in part by the National Science Foundation and by DARPA under the ACMP.