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PERIODIC POINTS OF THE BILLIARD BALL MAP IN A CONVEX DOMAIN

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Abstract

We study periodic orbits of the billiard ball map in a strictly convex domain with a C^∞ boundary. We conjecture that the Lebesgue measure of all periodic points is 0. We are able to prove the following partial result: the measure of period three periodic orbits is 0.

The question of whether the mentioned measure is 0 appeared in the study of spectral invariants of a planar region. The author learned about it from R. Melrose.

0. Introduction

In this paper we study the Lebesgue measure of the set of periodic points of a billiard ball map in a strictly convex region with a smooth boundary. The question of whether this measure is 0 has some significance in the theory of spectral invariants of a planar region (cf. [3]). We learned about the problem from R. Melrose. We notice that if the measure is 0 then the length spectrum of a billiard in a convex domain has measure 0.

In spite of our attempts to give a complete solution, we have only been able to settle the simplest case of period three. (The reader will notice that the case of period two is trivial since in this case the reflection has to happen under the right angle with the boundary.) The case of higher periods seems to be much more difficult for more or less the same reasons as verifying that a critical point of a multi-variable function is an extreme point in the case when the second derivative test fails. A solution along the same lines as in this paper seems likely to be related to the singularity theory of functions.

In our solution we use the length function $\mathscr{L}_n$ (see §1). It is the same function that Birkhoff used in his proof of the existence of many periodic orbits (cf. [3]).

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