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QUASICONFORMAL AND AFFINE GROUPS

GAVEN J. MARTIN

Introduction

Suppose that G is a discrete abelian group of diffeomorphisms acting on the unit sphere $\mathbf{S}^n$ of $\mathbf{R}^{n+1}$. The main result of this paper is that if G has uniformly bounded distortion and an element of infinite order, then G is conjugate, by a self-homeomorphism of $\mathbf{S}^n$ with bounded distortion, to a conformal group Γ of $\mathbf{S}^n$ (that is, Γ is a subgroup of the Möbius group). Actually, the restriction to abelian groups will be weakened to a class of admissible groups which will be defined by a simple algebraic condition (see §4). For instance, groups with an infinite cyclic central subgroup will be admissible; such groups can of course contain free groups of any rank. We will give a simple geometric condition on a subgroup of the euclidean group to be admissible. In [11], we showed that such a conjugacy exists in the case G is cocompact and isomorphic to a crystallographic group. Combining this with the results herein gives a wide class of abstract subgroups of the euclidean group for which any discrete and faithful representation in the diffeomorphism group of $\mathbf{S}^n$ with bounded distortion is conjugate into the euclidean group by a homeomorphism with bounded distortion.

We will provide a number of references from the recent literature to show how our results fit in with those obtained earlier. For instance, we recall from [11] that there is a uniformly quasiconformal group acting smoothly on $\mathbf{R}^n$ and isomorphic to a free abelian group of rank $n-1$ which is not quasiconformally conjugate to a euclidean group. Evidently it cannot be made smooth at infinity.

In order to study discrete groups of bounded distortion one is, of course, naturally led to the notion of a discrete quasiconformal group. For the basic facts regarding quasiconformal mappings we refer to Väisälä's book, [18] and for the theory of discrete quasiconformal groups we refer to the articles by Gehring and Martin [4] and Tukia [17].

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