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ON THE EVOLUTION OF HARMONIC MAPS IN HIGHER DIMENSIONS

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Abstract

We establish partial regularity results and the existence of global regular solutions to the evolution problem for harmonic maps with small data. The key ingredient is a decay estimate analogous to the well-known monotonicity formula for energy minimizing harmonic maps.

1. Let $\mathscr{M}, \mathscr{N}$ be (compact) Riemannian manifolds of dimensions m, n with metrics γ, g respectively. In local coordinates $x = (x^1, \cdots, x^m)$ and $u = (u^1, \cdots, u^n)$ we denote

$$\gamma = (\gamma_{\alpha\beta})_{1 \le \alpha, \beta \le m}, g = (g_{ij})_{1 \le i,j \le n} \text{ and } (\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}.$$

For a C^1-map $u\colon \mathscr{M} \to \mathscr{N}$ the energy of u is given by the intrinsic Dirichlet integral

$$E(u) = \int_{\mathscr{M}} e(u)\, d\mathscr{M}$$

with density

$$e(u; x) = \frac{1}{2}\gamma^{\alpha\beta}(x) g_{ij}(u) \frac{\partial}{\partial x^\alpha} u^i \cdot \frac{\partial}{\partial x^\beta} u^i$$

in local coordinates. A summation convention is used. Since $\mathscr{N}$ is compact, $\mathscr{N}$ may be isometrically embedded into $\mathbf{R}^N$ for some N, and E becomes the standard Dirichlet integral of maps $u\colon \mathscr{M} \to \mathscr{N} \subset \mathbf{R}^N$.

u is harmonic iff E is stationary at u; in particular

$$\begin{aligned} &\frac{d}{d\varepsilon} E(u + \varepsilon\phi)|_{\varepsilon=0} \\ &\quad = \int_U (-\Delta_{\mathscr{M}} u + \Gamma_{\mathscr{N}}(u)(\nabla u, \nabla u)_{\mathscr{M}})^i g_{ij}(u) \phi^j\, dx = 0 \end{aligned} \tag{1.1}$$

for any smooth variation ϕ with support in a coordinate neighborhood $U \subset \mathbf{R}^m$ and such that $(u + \varepsilon\phi)(U)$ is contained in a coordinate chart V in the target space, where

$$\Delta_{\mathscr{M}} = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta}\cdot\right)$$

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