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CAUSTICS AND EVOLUTES FOR CONVEX PLANAR DOMAINS

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Abstract

The caustics for the billiard ball map on an ellipse satisfy a certain evolution property. An operator relating the curvature of a caustic to the curvature of the boundary is defined for the billiard ball map in smooth convex planar domains and is used to derive an equation which characterizes those curves satisfying the evolution property as ellipses.

1. Introduction

Many dynamical systems have been modeled by a billiard ball travelling in a bounded domain in the plane. In considering this problem the objects of interest are periodic orbits—points on the boundary of the region to which a billiard ball returns after a fixed number of reflections—and invariant curves (caustics for optical reflection).

In an elliptic domain, a billiard ball whose trajectory is tangent to an ellipse confocal with the boundary returns to the inside ellipse and its trajectory is again tangential to the inside ellipse, so confocal ellipses define invariant curves and the billiard system on an ellipse is integrable. Birkhoff conjectured that the only integrable convex planar regions with smooth boundaries are ellipses. Seemingly in contrast, it was shown by Moser that in any planar region with a sufficiently smooth boundary, the billiard ball map has enough invariant curves so that the Lebesgue measure of their associated rotation numbers is positive [6] (see also [3]).

What seems to distinguish ellipses from other smooth curves is the *evolution property*—the caustics for an elliptic region are themselves integrable and share their caustics with the elliptic boundary. This property is perhaps easier to see from the point of view of a caustic. If a curve inside a domain is a caustic for the billiard ball map on the domain's boundary, we call the domain's boundary the evolute of the curve. In plane geometry (see Lemma 1), if we loop a string around a closed curve, lean a pen against the string, and draw, then we describe the evolute of the curve. The evolution property

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