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SO(3)-CONNECTIONS AND RATIONAL HOMOLOGY COBORDISMS

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1. Introduction

The main purpose of this paper is the study of rational homology cobordisms of rational homology 3-spheres. In particular it is shown that the ρ_α-invariants of Atiyah-Patodi-Singer [2] which can be defined as spectral invariants are, under some extra conditions, integral homology cobordism invariants of rational homology spheres.

Related to this study is the question of when a rational homology sphere Σ bounds a rational homology ball. This can also be answered in some cases in terms of ρ_α-invariants. In turn, these invariants can then be used to answer questions concerning sliceness of knots. Casson and Gordon [3] have constructed an invariant that detects when a two-bridge knot is not ribbon. This invariant is actually the ρ_α-invariant for the double branched cover Σ of S^3 branched over $K \subset S^3$ and character $\alpha: H_1(\Sigma) \to \mathrm{U}(1)$. For characters of prime power order Casson and Gordon [3] also show that this is a slice invariant. Namely, if $\rho_\alpha(\Sigma) = \sigma(K, \alpha) \neq \pm 1$ then K is not ribbon, and if α is of prime power order they can also conclude that it is not slice. In the case that Σ is a spherical space form, Fintushel and Stern [7] remove the condition that α be a prime-power order. In this paper the condition that Σ be a spherical space form is replaced by a weaker condition that $H^1(\Sigma, \mathbf{L}_\alpha) = 0$, where $\mathbf{L}_\alpha$ is the flat complex line bundle induced by the character α.

More specifically, let X be a compact, smooth 4-manifold with boundary ∂X. Let α: $\pi_1(X) \to \mathrm{U}(1)$ be a nontrivial character. It defines a complex line bundle $\mathbf{L}_\alpha$ on X by $\mathbf{L}_\alpha = \tilde{X} \times_\alpha \mathbf{C}$ where $\tilde{X}$ is the universal cover of W. Let $\operatorname{sign}_\alpha(X)$ denote the signature of the hermitian form induced on $H^2(X, \mathbf{L}_\alpha)$ by the cup-product. Then $\rho_\alpha(\partial X) = \operatorname{sign}(X) - \operatorname{sign}_\alpha(X)$ is a differential invariant of the boundary [2].

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